IXth Oporto Meeting on Geometry, Topology and Physics

GEOMETRY IN DIMENSIONS 6, 7 AND 8

Simon Salamon

Advertised programme

Lecture 1

SU(3) holonomy and Calabi-Yau manifolds
The 6-sphere and nearly-Kähler spaces
Complex structures defined by 3-forms

Lecture 2

G_2 structures defined by self-duality
Explicit metrics with holonomy G_2
Introduction to Joyce's examples

Lecture 3

Index theory for holonomy $Sp(2), SU(4), Spin\,7$
Yang-Mills theory on Wolf spaces
Compact hyperkähler manifolds

LECTURE 1

Complex structures on 6-manifolds

Example The complex Heisenberg group G consists of $(z^1,z^2,z^3) = \begin{pmatrix} 1 & z^1 & z^3 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{pmatrix}$ with

$$(u^1,u^2,u^3)\cdot(z^1,z^2,z^3) = (u^1+z^1,\ u^2+z^2,\ u^3+z^3+u^1z^2).$$

It is a complex Lie group; if $L_{\mathbf{u}}$ denotes left multiplication by $\mathbf{u} = (u^1,u^2,u^3)$ then

$$\begin{cases} L_{\mathbf{u}}^*(dz^r) = d(u^r+z^r), \\ L_{\mathbf{u}}^*(dz^3 - z^1dz^2) = d(u^3+z^3+u^1z^2 - (u^1{+}z^1)d(u^2{+}z^2)) = dz^3 - z^1dz^2. \end{cases}$$

Let

$$dz^1 = e^1 + ie^2, \quad dz^2 = e^3 + ie^4, \quad -dz^3 + z^1dz^2 = e^5 + ie^6.$$

Then $\{e^1,\dots,e^6\}$ forms a basis of $\mathfrak{g}^*$, the space of real left-invariant 1-forms, satisfying

$$de^r = \begin{cases} 0, & 1 \leqslant i \leqslant 4, \\ e^{13} + e^{42}, & i = 5, \\ e^{14} + e^{23}, & i = 6. \end{cases}$$

These relations determine the Lie bracket on $\mathfrak{g}$ by the rule $e^r([u,v]) = -de^r(u,v)$ and we obtain a complex

$$\mathfrak{g}^* \xrightarrow{d=d_1} \textstyle\bigwedge^2\mathfrak{g}^* \xrightarrow{d_2} \textstyle\bigwedge^3\mathfrak{g}^* \xrightarrow{d_3} \cdots$$

Consider the subgroup

$$\Gamma = \{(z^1, z^2, z^3) \in G : z^r = a^r + ib^r \in \mathbb{Z}[i]\}.$$

The Iwasawa manifold $N = \Gamma\backslash G$ (set of right cosets) is the total space of a principal bundle $\pi : N \to T^4$ with fibre T^2, where π is induced from $(z^1, z^2, z^3) \mapsto (z^1, z^2)$. The 1-forms e^i are well defined on N, and

$$\mathbb{D} = \langle e^1, e^2, e^3, e^4 \rangle = \pi^*(T_x^* T^4).$$

Observe that

$$d(\mathfrak{g}^*) \subset \bigwedge{}^{+}\mathbb{D} \subset \bigwedge{}^{2}\mathbb{D},$$

where $\bigwedge^{+}\mathbb{D}$ is the 3-dimensional subspace of 'self-dual' 2-forms (relative to the obvious inner product). This is one of many examples that illustrates the role that self-duality plays in higher-dimensional geometry.

<u>Question</u> What algebraic structure does the bi-invariant complex structure of G (that passes to N) impose upon $\mathfrak{g}^*$?

Familiar answer: an endomorphism $J : \mathfrak{g}^* \to \mathfrak{g}^*$ with $Je^1 = -e^2$ etc, so that $Jdz^r = i\,dz^r$, and satisfying $[JX, JY] = [X, Y] + [JX, Y] + [X, JY]$.

Less familiar answer: a real 2-dimensional space of closed 3-forms, spanned by

$$\begin{cases} \phi = (e^{13} + e^{42})e^5 - (e^{14} + e^{23})e^6, \\ \psi = (e^{13} + e^{42})e^6 + (e^{14} + e^{23})e^5, \end{cases}$$

where $\phi + i\psi = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6) = \phi + i\psi$.

Lemma[1] $\{g^*\phi : g \in GL(6,\mathbb{R})\}$ is an open set $\mathcal{O}$ of $\bigwedge^3\mathbb{R}^6$. Indeed, the stabilizer of ϕ is $SL(3,\mathbb{C})$, so $\mathcal{O} \cong GL(6,\mathbb{R})/SL(3,\mathbb{C})$.

Given $\phi \in \mathcal{O}$, there is therefore a unique pair $(\widehat{\phi}, J)$ where $\widehat{\phi} \in \mathcal{O}$ and J is an oriented almost complex structure on $\mathbb{R}^6$ such that $\phi + i\widehat{\phi} \in \Lambda^{3,0} \oplus \Lambda^{0,3}$.

Regard $\mathcal{O}$ as a subset of $\bigwedge^3\mathfrak{g}^*$. If $d\phi = 0 = d\widehat{\phi}$ then J is integrable. The space $\mathcal{C}^+(\mathfrak{g})$ of invariant oriented complex structures on N is isomorphic to $\Sigma/\mathbb{C}^*$, where

$$\Sigma = \{\phi \in \ker d_3 \cap \mathcal{O} : d\widehat{\phi} = 0\}.$$

In fact, $\dim(\ker d_3) = 15$ and Σ is a real cubic hypersurface, so $\mathcal{C}^+(\mathfrak{g})$ has real dimension 12.

Theorem[2] $\mathcal{C}^+(\mathfrak{g})$ has the homotopy type of the disjoint union of a point and a 2-sphere S^2.

The treatment of symplectic structures is somewhat easier:

Proposition The space $\mathcal{S}^+(\mathfrak{g})$ of invariant symplectic forms on N has real dimension 10, and is homotopic to S^3.

Idea. First observe that N admits symplectic forms, like $e^{16}+e^{25}+e^{34}$. One can also take $\omega = f \wedge e^6 + \cdots$ for any $f \in S^3 \subset \mathbb{D}$, the rest of ω being uniquely determined by the orientation and the fact that $d\omega = 0$.

There is a sense in which S^3 separates the point and S^2 of the theorem.

[1] Recently exploited by Hitchin in a study of Calabi-Yau moduli spaces.

[2] Proved by G. Ketsetzis, using fact that $\mathbb{D}$ is always J-invariant, and extending work of Abbena-Garbiero-Salamon.

The Kähler condition

Theorem[3] A nilmanifold (like N) cannot admit a Kähler metric unless it is a torus.

Recall that a Riemannian metric g is Kähler if there exists an orthogonal complex structure J for which $d\omega = 0$, where $\omega_{bc} = J_b^a g_{ac}$. This implies that $\nabla J = 0$, and parallel transport preserves not only g but J and ω: the holonomy group is contained in $U(n)$.

Definition A Calabi-Yau manifold is a compact Kähler manifold with holonomy group equal to $SU(n)$.

This means that there is a parallel $(n,0)$-form $\phi + i\psi$, and that the algebra of parallel forms is generated by ω, ϕ, ψ.

Let M be a compact Kähler manifold of real dimension 6, with a nowhere-zero closed $(3,0)$ form Φ. Then the canonical bundle $\Lambda^{3,0}$ is trivial, $c_1(M) = c_1(TM) = -c_1(\Lambda^{3,0})$ vanishes in $H^2(M,\mathbb{R})$. Yau's Theorem implies that M has a Ricci-flat Kähler metric, and it can then be shown that $\nabla\Phi = 0$.

Proposition Such an M is projective, i.e. a submanifold of some $\mathbb{CP}^m$.

Relies on the fact that $H^{2,0} = 0$. The cone of Kähler forms on (M,J) is open in $H^2(M,\mathbb{R}) = H^{1,1}$ and so intersects $H^2(M,\mathbb{Z}) \cong H^1(M,\mathcal{O}^*)$. The result follows from the Kodaira embedding theorem.

[3] Proofs by Hano, Benson-Gordon, McDuff, Campana and others.

Two constructions

1. The intersection of two hypersurfaces Σ_1, Σ_2 in $\mathbb{CP}^5$ defined by polynomials f_1, f_2 of degrees d_1, d_2. If $df_1 \wedge df_2 \neq 0$ at all points of $M = \Sigma_1 \cap \Sigma_2$ then M is a complex manifold.

$$T\mathbb{CP}^5|_M \cong TM \oplus \mathcal{O}(d_1) \oplus \mathcal{O}(d_2),$$

and $6 = c_1(TM) + d_1 + d_2$. So $c_1 = 0$, (d_1, d_2) is one of $(1,5), (2,4), (3,3)$. The first case gives a quintic in $\mathbb{CP}^4$, and $\chi = c_3 = -200, -176, -144$ respectively.[4]

2. An example more akin to a Kummer surface[5]: Let $\varepsilon = (-1 + \sqrt{3})/2$ be a primitive cube root of unity, and set

$$\Gamma = \{(z^1, z^2, z^3) : z^r = a^r + \varepsilon b^r \in \mathbb{Z}[\varepsilon]\}.$$

Then $\Gamma \backslash (\mathbb{C}^3, +)$ is diffeomorphic to T^6. Multiplication by ε on $\mathbb{C}^3$ induces a mapping $\theta : T^6 \to T^6$ with $\theta^3 = 1$ that preserves the 3-form $dz^1 \wedge dz^2 \wedge dz^3$. Then θ has 27 fixed points, and $T^6/\langle\theta\rangle$ has 27 singular points locally resembling $\mathbb{C}^3/\mathbb{Z}_3$.

Now $\mathbb{C}^3 \setminus \{0\}$ can be identified with the total space of the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{CP}^2$ minus its zero section. It follows that the total space $\Lambda^{2,0}$ of $\mathcal{O}(-3) \to \mathbb{CP}^2$ admits a mapping π to $\mathbb{C}^3/\mathbb{Z}_3$ such that $\pi^{-1}(0) \cong \mathbb{CP}^2$. Since $\Lambda^{2,0}$ also has a canonical $(3,0)$-form (d of the tautological $(2,0)$-form), there exists an overall resolution X of $T^6/\langle\theta\rangle$, with both a Kähler metric and nowehere-zero holomorphic $(3,0)$-form. Thus, X has a Calabi-Yau metric. Since $b_2 = 9 + 27$ and $b_3 = 2$, X has $h^{1,1} = 0$ (bad news for the 'mirror' of X) and $\chi = 76$.

[4] Thousands of distinct CY manifolds have been enumerated by various generalizations of this construction.
[5] Due to Roan.

LECTURE 2

G_2 Structures on real 7-manifolds

On $\mathbb{R}^6$ there is a standard

- inner product $g = \sum_1^6 e^r \otimes e^r$
- 2-form $\omega = \sum_1^6 Je^r \otimes e^r = 12 + 34 + 56$
- 3-form $\phi = (13 + 42)5 - (14 + 23)6$

Consider

$$\varphi = \phi - \omega \wedge e^7 = -(12 + 34)7 + (13 + 42)5 - (14 + 23)6 - 567.$$

This form is invariant by subgroups $SU(3)$ and (a non-standard) $SO(4)$ of $SO(7)$.

Proposition $\mathcal{O} = \{g^*\varphi : g \in GL(7, \mathbb{R})\}$ is an open set of $\bigwedge^3 \mathbb{R}^7$. In fact, $\mathrm{Stab}(\varphi) = G_2 \subset SO(7)$ has dimension 14 and $\mathcal{O} \cong GL(7, \mathbb{R})/G_2$.

Corollary Let φ be a 'positive' 3-form on a 7-manifold M, meaning that $g \in \mathcal{O} \subset \bigwedge^3 T^*M$ at each point. Then φ determines a metric g and also a 4-form $\widehat{\varphi} = *_g \varphi$. If $d\varphi = 0 = d\widehat{\varphi}$ then $\nabla\varphi = 0$ and g has holonomy in G_2 [FG]. This is because

$$\begin{cases} \bigwedge^2 T^* \cong \mathfrak{so}(7) \cong \mathfrak{g}_2 \oplus T^* \\ \bigwedge^3 T^* \cong \mathbb{R}\varphi \oplus T^* \oplus S^2_0 T^* \end{cases} \quad \text{whereas} \quad \nabla_X \varphi \in T^* \text{ and } \nabla\varphi \in T^* \otimes T^*$$

If $\nabla\varphi = 0$, the Riemann curvature tensor R belongs to the kernel of the linear mapping $S^2(\mathfrak{g}_2) \to \bigwedge^4 T^*$ (given by wedging 2-forms), which is an irreducible 77-dimensional subspace.[6] Hence g is Ricci-flat.

[6] Beware that there are two inequivalent 77-dimensional irreducible representations of $\mathfrak{g}_2$; the other is the kernel of the Lie bracket $\bigwedge^2 \mathfrak{g}_2 \to \mathfrak{g}_2$.

Example $M = S^3 \times S^3$ has a basis of 1-forms $\{\underbrace{e^1, e^3, e^5}, \underbrace{e^2, e^4, e^6}\}$ such that $de^1 = e^{35}$, $e^3 = e^{51}$, $de^5 = e^{13}$ etc. Consider

$$i(e^1 + \varepsilon e^2) \wedge (e^3 + \varepsilon e^4) \wedge (e^5 + \varepsilon e^6) = \phi + i\psi.$$

Multiplication by ε on $T_c M$ arises from the fact that $G \times G = \dfrac{G \times G \times G}{G}$ is a 3-symmetric space.[7]

Lemma $d\omega = -\frac{2}{\sqrt{3}}\phi$ (so $d\phi = 0$) and $d\psi = \frac{1}{2}\omega^2$.

These equations characterize a 'nearly-Kähler' (NK) metric on a 6-manifold M. Only other compact examples known are S^6, $\mathbb{CP}^3$, $\mathbb{F}^3 = \dfrac{SU(3)}{T^2}$.

Theorem[8] If M^6 is NK then $M \times \mathbb{R}^+$ has holonomy in G_2.

Follows because we may construct closed forms

$$\begin{aligned} \varphi &= 4t^{1/2}\phi - \sqrt{3}t^{-1/2}\omega \wedge dt, \\ *\varphi &= t^{-1/2}\psi \wedge dt - \tfrac{1}{2}t^{1/2}\omega^2 \end{aligned}$$

$M = S^6$ gives the flat metric on $\mathbb{R}^7$; other conical metrics can be deformed into complete metrics with holonomy equal to G_2, analogous to the Eguchi-Hanson gravitational instanton[9].

E.g.

$$(r+1)^{1/2}\pi^* g_{S^4} + (r+1)^{-1/2} g_{\mathbb{R}^3} \quad \text{on total space of} \quad \begin{array}{l} \bigwedge^+ T^* S^4 \supset \mathbb{CP}^3 \\ \downarrow \pi \\ S^4 \end{array}$$

[7] The full theory of these was developed by Gray and Wolf.
[8] Proved by R. Reyes-Carrion, present in Porto.
[9] A hyperkähler metric on the total space of $\mathcal{O}(-2) \to \mathbb{CP}^1$, also described by Calabi

Cohomology of compact G_2-manifolds

Let φ be a positive 3-form on a compact 7-manifold M with $d\varphi = 0 = d\widehat{\varphi}$.

- Berger's list implies that the holonomy equals G_2 iff $b_1 = 0$

- $H^2(M, \mathbb{R}) = \{\beta \in \Gamma(\Lambda^2_{14}) : d\beta = 0\}$ and

$$8\pi^2 p_1 \cup [\varphi] = \int_M \text{trace}(R \wedge R) \wedge \varphi = \|R\|^2.$$

Here Λ^2_{14} is shorthand for the distinguished subbundle of 2-forms of rank 14 (and fibre isomorphic to $\mathfrak{g}_2$).

<u>Corollary</u> No compact 7-manifold with $0 = p_1 \in H^4(M, \mathbb{R})$ (e.g. a nilmanifold which is parallelizable) can have a metric with holonomy equal to G_2.

- $V := H^3(M, \mathbb{R}) = \mathbb{R}[\varphi] \oplus \{\gamma \in \Gamma(\Lambda^3_{27}) : d\gamma = 0\}$ represents the tangent space to the space of G_2-structures modulo diffeomorphism.[10]

A real-valued function on V is defined by $f([\varphi]) = \int_M \varphi \wedge \widehat{\varphi}$ and the 1-form $df : V \to V^* = H^4(M, \mathbb{R})$ maps $[\varphi]$ to $[\widehat{\varphi}]$.

<u>Corollary</u> The set $\{([\varphi], [\widehat{\varphi}]) : \varphi \text{ is a positive 3-form}\}$ is (the graph of df and so) a Lagrangian submanifold of dimension b_3 of the symplectic space $V \oplus V^* \cong T^*V$.

Solving $d\varphi = 0 = d\widehat{\varphi}$ for $\varphi = \varphi_0 + \xi + d\eta$ with gauge fixing condition $d^*\varphi \in \Gamma(\Lambda^2_{14})$ is equivalent to an elliptic non-linear PDE $(dd^* + d^*d)\eta = *dF(\xi + d\eta)$, where F is a smooth mapping of a neighbourhood of 0 in $\bigwedge^3 T^*_x M$ to $\bigwedge^4 T^*_x M$ with $F(0) = 0$.

[10] Work of Bryant and Joyce.

<u>One of Joyce's examples</u>[11] Let $T^7 = \dfrac{\mathbb{C}^3 \oplus \mathbb{R}}{\Gamma \oplus \mathbb{Z}}$, where $\Gamma = (\mathbb{Z} \oplus \varepsilon\mathbb{Z})^3$.

Define $\alpha, \beta, \gamma : T^7 \to T^7$ by

$$\begin{aligned} \alpha(z^1, z^2, z^3, t) &= (\varepsilon z^1, \varepsilon^2 z^2, z^3, t) \\ \beta(z^1, z^2, z^3, t) &= (z^1, \varepsilon z^2, \varepsilon^2 z^3, t + \tfrac{1}{3}) \\ \gamma(z^1, z^2, z^3, t) &= (\overline{z}^1, \overline{z}^2, \overline{z}^3, -t) \end{aligned}$$

$\langle \alpha, \beta \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, and $\Delta = \langle \alpha, \beta, \gamma \rangle$ is a non-abelian group of order 18. Δ fixes the flat G_2-structure, but not the SU(3) structure on $\mathbb{C}^3$, so any desingularization preserving the G_2 structure will then lead to a manifold with $\mathrm{Hol}{=}G_2$.

<u>Lemma</u> T^7/Δ has singular set consisting of (i) $T^3/\mathfrak{S}_3$ (with $z^1{=}z^2{=}0$), (ii) four manifolds $T^3/\mathbb{Z}_3$, (iii) two T^3 each intersecting $T^3/\mathfrak{S}_3$ in T^1.

Resolving each type, e.g. in (ii) by replacing $T^3 \times (\mathbb{C}^2/\mathbb{Z}_3)$ by $T^3 \times$ ALE with $b_2(\text{ALE}){=}2$,

$$(b_2, b_3) \mapsto \begin{cases} (b_2{+}1, b_3{+}1) \text{ or } (b_2, b_3{+}1) & \text{for (i)} \\ (b_2{+}2, b_3{+}2) \text{ or } (b_2, b_3{+}2) & \text{for (ii)} \quad \text{giving } (0{+}k{+}2,\ 4{+}15) \\ (b_2{+}1, b_3{+}3) & \text{for (iii)} \quad \text{overall, with } 0 \leqslant k \leqslant 9 \end{cases}$$

Joyce's exhaustive analysis has furnished 252 other sets of Betti numbers (b_2, b_3) with $0 \leqslant b_2 \leqslant 28$ and $4 \leqslant b_3 \leqslant 215$. The vast majority have $b_2 + b_3 \equiv 3 \bmod 4$.

[11] One of the more complicated ones in D. Joyce's book 'Compact manifolds with special holonomy' (OUP, 2000) which incorporates sweeping generalizations of constructions in his previous JDG papers on the subject. See page 338, opposite the graph of possible Betti numbers.

LECTURE 3

Special Riemannian 8-manifolds

Let (M^8, g) be a simply-connected Riemannian 8-manifold which is irreducible (not $M_1 \times M_2, g_1 \times g_2$).

<u>Theorem</u> The holonomy (parallel transport) group Hol is one of $SO(8)$, $U(4)$, $SU(4)$, $Sp(2)$, $Spin\,7$, $Sp(2)Sp(1)$, $U(2)Sp(1)$, $SU(2)Sp(1)$, $SU(3)$.[12]

- $\mathrm{Hol} \subseteq U(4) \;\Rightarrow\;$ M is Kähler and has a parallel form $\omega = \omega_1 \in \Lambda^{1,1}$, pointwise of type $12+34+56+78$

- $\mathrm{Hol} = SU(4) \;\Rightarrow\;$ M also has a parallel form $\phi + i\psi \in \Lambda^{4,0}$, pointwise $(1+i2)(3+i4)(5+i6)(7+i8)$
  $\Rightarrow\;$ M is Ricci-flat Kähler (and CY)

We have already discussed these reductions. The next on the list is

- $\mathrm{Hol} = Sp(2) \;\Rightarrow\;$ M has a parallel holomorphic symplectic form $\omega_2 + i\omega_3 \in \Lambda^{2,0}$, pointwise $(13+42+25+86) + i(14+23+58+67)$
  $\Rightarrow\;$ M is hyperkähler (HK)

On any HK manifold, the 2-forms $\omega_1, \omega_2, \omega_3$ determine respective complex structures J_1, J_2, J_3 (with $J_i e^1 = -e^i$ etc) satisfying the quaterion identity $J_1 J_2 = J_3 = -J_2 J_1$.

[12]The last space was omitted in error in the lecture, as the speaker forgot that any compact Lie group is itself a symmetric space. The theorem is then a special case of the celebrated classification results of E. Cartan, and Berger's list.

- $Sp(2)Sp(1)$ is the stabilizer of $\omega_1^2+\omega_2^2+\omega_3^2\in\bigwedge^4\mathbb{R}^8$
  If $\mathrm{Hol}\subseteq Sp(2)Sp(1)$ then M is quaternion-Kähler (QK); e.g. $\mathbb{HP}^2$.
  Such manifolds do not generally admit globally-defined complex structures.

- $Spin\,7$ is the stabilizer[13] of the 4-form $\omega_1^2+\omega_2^2-\omega_3^2$ in $\mathbb{R}^8$
  If $\mathrm{Hol}\subseteq Spin\,7$ then $\mathrm{Ricci}=0$. The theory of these manifolds has many other similarities with those with $\mathrm{Hol}=G_2$.

- The last three in list are holonomy groups of symmetric spaces. Up to local isometry,

  $$\mathrm{Hol}=U(2)Sp(1)\Rightarrow M=\frac{SU(4)}{U(2)Sp(1)}=\mathbb{G}\mathrm{r}_2(\mathbb{C}^4)=\mathbb{G}\mathrm{r}_4(\mathbb{R}^6)=Q^4\subset\mathbb{CP}^5$$

  $$\mathrm{Hol}=SU(2)Sp(1)\Rightarrow M=\frac{G_2}{SU(2)Sp(1)}=\frac{G_2}{SO(4)}$$

  $\mathrm{Hol}=SU(3)\Rightarrow M$ is itself the Lie group $SU(3)=\dfrac{SU(3)\times SU(3)}{SU(3)}$

Any QK 8-manifold with $\mathrm{Ricci}>0$ is one of the three spaces above.

<u>Wolf's theorem</u> For any compact simple G, there is

- a QK symmetric space $\dfrac{G}{K\,Sp(1)}$
- an associated 'adjoint variety' $\dfrac{G}{K\,U(1)}=\dfrac{\text{minimal nilpotent orbit in }\mathfrak{g}_c}{\mathbb{C}^*}$

The latter is a Fano contact manifold.[14]

[13] Observation of Bryant-Harvey.
[14] The QK space may be regarded as an $\mathbb{H}^*$-quotient of the nilpotent orbit, which itself has a HK metric described by Kronheimer.

Curiosities from representation theory

The subgroups $U(2)$ and $SU(2)$ of $Sp(2)$ themselves give rise to

$\dfrac{Sp(2)}{U(2)} = \dfrac{SO(5)}{SO(2)\times SO(3)} = \mathbb{G}\mathrm{r}_2(\mathbb{R}^5) = Q^3 \subset \mathbb{CP}^4,$

$\dfrac{Sp(2)}{SU(2)} = \dfrac{SO(5)}{SO(3)} = B,$ with T_xB the irreducible 7-dim rep V_7 of $SO(3)$

$\dfrac{SO(8)}{SU(3)}$ is a 20-dim space with a non-integrable almost complex structure

It can be shown[15] that

$$\bigwedge^3 V_7 \cong V_{13} \oplus V_9 \oplus V_7 \oplus V_5 \oplus V_1$$

It follows that T_xB has a unique 3- (and 4-) form invariant by $SO(3)$. The unique $SO(5)$-invariant metric on B has strictly positive sectional curvature.[16]

Corollary B has a G_2 structure with $d\varphi = *\varphi$, and (by analogy to the NK case), $B\times\mathbb{R}^+$ a metric with $\mathrm{Hol} = Spin\,7$.

The homomorphism $SO(3) \to G_2 \subset SO(V_7)$ itself gives a description

$$\mathfrak{g}_2 = \mathfrak{so}(3) \oplus V_{11} = V_3 \oplus V_{11}$$

entirely in terms of representations of $SO(3)$ or $SU(2) = Sp(1)$!

[15] The book by Fulton-Harris on Representation Theory contains examples of this type.
[16] Proved by Berger, hence the notation.

Poincaré polynomials $1+b_1t+b_2t^2+\cdots$

	$\mathbb{CP}^3$, Q^3 :	$1+t^2+t^4+t^6$
	S^7, B :	$1+t^7$
QK	$\mathbb{HP}^2$, $G_2/SO(4)$:	$1+t^4+t^8$
QK	$\mathbb{G}\mathrm{r}_2(\mathbb{C}^4)=\mathbb{G}\mathrm{r}_4(\mathbb{R}^6)$:	$1+t^2+2t^4+t^6+t^8$
	$b_4=1+b_2$	
HK	$\mathbb{K}^{[2]}$:	$1+23t^2+276t^4+23t^6+t^8$
HK	$\mathbb{K}_2$:	$1+7t^2+8t^3+108t^4+8t^5+7t^6+t^8$
	$b_3+b_4=46+10b_2$	

The quadric Q^{2n+1} has the same Betti numbers as $\mathbb{CP}^{2n+1}$. The spaces B and $G_2/SO(4)$ mimic S^7 and $\mathbb{HP}^2$ respectively, but have torsion classes.

Theorem The Betti numbers of compact QK (with $\mathrm{Ricci}>0$) and HK 8-manifolds satisfy the highlighted constraints.[17]

[17] The former is an essential ingredient in Poon-Salamon's proof that there are no other positive QK 8-manifolds

Let $\mathbb{T} = T^4 = \mathbb{R}^4/\mathbb{Z}^4$, and $\mathbb{K}$ a K3 surface formed by resolving $\mathbb{T}/\pm 1$.

The dihedral group $D = \{e, \alpha, \alpha^2, \alpha^3, \beta, \alpha\beta, \alpha^2\beta, \alpha^3\beta\}$ generated by

$$\alpha(z^1, z^2, z^3, z^4) = (-z^3, -z^4, z^1, z^2), \quad \beta(z^1, z^2, z^3, z^4) = (z^3, z^4, z^1, z^2)$$

acts on $T^8 = \mathbb{T} \times \mathbb{T}$.

Then $\mathbb{K}^{[2]} := \mathrm{Hilb}^2\mathbb{K}$ is a resolution of $\dfrac{\mathbb{K} \times \mathbb{K}}{\mathfrak{S}_2}$, or equivalently of $\dfrac{\mathbb{T} \times \mathbb{T}}{D}$ which has 17 singular $\mathbb{T}/\pm 1$ (which explains why $b_2(\mathbb{K}^{[2]}) = b_2(\mathbb{T}) + 17$[18]).

$$\begin{array}{ccc}
\mathbb{T} \times \mathbb{T} & \longleftrightarrow & \{e\} \\[2ex]
\mathbb{T}^{[2]} \ (\supset \mathbb{K}) & \longleftrightarrow & \langle \beta \rangle \\[2ex]
? & \longleftrightarrow & \langle \alpha \rangle \\[4ex]
\mathbb{K}^{[2]} & \longleftrightarrow & D
\end{array}$$

In this 'Galois correspondence', spaces on the left are HK resolutions of $\mathbb{T} \times \mathbb{T}$ factored out by the indicated subgroup. $\mathbb{T}^{[2]}$ is a finite quotient of $\mathbb{T} \times \mathbb{K}$ and contains $\mathbb{K}$ as a submanifold.

It is impossible to fill the '?' since $\dfrac{\mathbb{T} \times \mathbb{T}}{\langle \alpha \rangle}$ has isolated singularities of local type $\mathbb{C}^4/\mathbb{Z}_2$ that admit no resolution with a HK metric.

[18] The Betti numbers of $\mathbb{K}^{[n]}$ were computed by Göttsche.

An application of index theory

Theorem

$$\hat{A} = -\frac{1}{24}(1 + b_2 - b_3 - b_4^+ + 2b_4^-) = \begin{cases} 0 & \text{if } M \text{ is QK} \\ 1 & \text{if } \mathrm{Hol} = Spin\,7 \\ 2 & \text{if } M \text{ is CY} \\ 3 & \text{if } M \text{ is HK} \end{cases}$$

Proof. Uses Pontrjagin classes $p_1 \in H^4(M, \mathbb{R})$, $p_2 \in H^8(M, \mathbb{R})$.

$$\begin{cases} 2-2b_1+2b_2-2b_3+b_4 = \chi = \frac{1}{8}(4p_2 - p_1^2) \\ b_4^+ - b_4^- = \sigma = \frac{1}{45}(7p_2 - p_1^2) \quad \text{[Hirzebruch]} \\ \text{number of } \| \text{ spinors} = \hat{A} = \frac{1}{5760}(7p_1^2 - 4p_2) \quad \text{[Atiyah-Singer]} \end{cases}$$

The Betti number constraints follow from additional vanishing-type arguments, namely $b_3 = b_4^- = 0$ in the QK case, and $b_2 = 3 + p$ and $b_4^- = 3p$ for HK. Any compact 8-dimensional HK manifold satisfies $\chi \leqslant 324$.[19]

Three big problems in the area of special metrics are

(i) to classify compact 6-dimensional NK manifolds;

(ii) to prove that all positive QK manifolds are symmetric;

(iii) to find other irreducible HK 8-manifolds.

[19] As observed by Beauville and Guan, this follows from a theorem of Verbitski.

Instantons on the quaternionic plane

$\mathbb{H}^3 = \mathbb{C}^6$ (with $\omega = 12+34+56$) is a representation of $Sp(3,\mathbb{C})$, whose adjoint variety is the total space of the Hopf fibration $\mathbb{CP}^5 \to \mathbb{HP}^2$.

Let H be tautological bundle over $\mathbb{HP}^2$ with fibre $\mathbb{C}^2$, so

$$\mathbb{C}^6 = H_x \oplus H_x^\perp$$

for each $x \in \mathbb{HP}^2$. Then $H^\perp$ is an 'instanton': it has a 'self-dual' connection[20] which makes $\pi^* H^\perp$ a standard holomorphic rank 4 bundle over $\mathbb{CP}^5$.

If $\phi = (13+42)5 - (14+23)6$ then $\omega \wedge \phi = 0$ and for each $x \in \mathbb{HP}^2$,

$$\phi \in {\textstyle\bigwedge}_0^3 \mathbb{C}^6 \longrightarrow H_x \otimes {\textstyle\bigwedge}_0^2 H_x^\perp \cong \mathrm{Hom}(H_x, {\textstyle\bigwedge}_0^2 H_x^\perp)$$

Theorem[21] (i) $V = \mathrm{coker}\phi$ is a complex rank 3 vector bundle on $\mathbb{HP}^2$, with structure group $SU(3)$;

(ii) $\phi \in \Gamma(\mathbb{HP}^2, H \otimes {\textstyle\bigwedge}_0^2 H^\perp)$ is a solution of a 'twistor equation';

(iii) $\pi^* V$ is an indecomposable holomorphic vector bundle on $\mathbb{CP}^5$, first discovered by Horrocks.

Condition (i) is a consequence of the generic nature of the 3-form ϕ that was discussed in the first lecture, and the theorem is another illustration of its importance.

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[20] The defining condition is that curvature takes values in the subspace of ${\textstyle\bigwedge}^2 T_x^* \mathbb{HP}^2$ isomorphic to $\mathfrak{sp}(2)$ generalizing ${\textstyle\bigwedge}^+$, and this is equivalent to Tian's condition involving the 4-form. Such connections are minima of the Yang-Mills functional. In fact H has curvature in $\mathfrak{sp}(1)$, and $H^\perp$ in $\mathfrak{sp}(2)$.

[21] Proved by Mamone Capria-Salamon.