Exceptional and hyperkähler geometry

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We introduce the concept of a G_2-structure, as a starting point for the construction of T^2 invariant Ricci-flat metrics in 7 dimensions. The underlying geometry brings together symplectic forms and quotients, hyperkähler metrics and Monge-Ampere type equations.

Introduction

$$S^6 = \frac{G_2}{SU(3)} \cong \frac{SO(7)}{SO(6)}$$

Interested in Riemannian 7-manifolds (X, g) whose structure reduces to G_2. This is detected by a certain 3-form φ at each point $x \in X$ or by the 'dual' 4-form φ^*.

In the simplest case $X = N^6 \times S^1$ with $\eta = dt$ on S^6,

$$\boxed{\begin{aligned} \varphi &= \sigma \wedge \eta + \psi_+ \\ \varphi^* &= \psi_- \wedge \eta + \tfrac{1}{2}\sigma \wedge \sigma \end{aligned}}$$

where σ is a non-degenerate 2-form on N and $\Omega = \psi_+ + i\psi_-$ a complex volume 3-form.

The G_2-structure on X is said to be torsion-free if $\mathrm{Hol}(g) \subseteq G_2$. This occurs iff φ and φ^* are *both* closed. In the product case this means

$$d\sigma = 0, \quad d\Omega = 0$$

and N has holonomy in $SU(3)$ ('Calabi-Yau').

Hypersurfaces and quotients

Aim is to construct torsion-free G_2 structures on 7-manifolds.

The resulting metrics are automatically Ricci-flat: $\boxed{R_{ij} = 0}$

Let (X, g) be such a structure, with $\mathrm{Hol}(g) = G_2$.

Reductions to $SU(3)$ can still be accomplished pointwise in the following situations:

$$\begin{array}{ccc} H^6 & \hookrightarrow & X \\ & & \Big\downarrow S^1 \\ & & Q^6 \end{array}$$

How far are H, Q from being CY?

Assume the circle action preserves φ so that $\sigma = V \lrcorner \varphi$ is a symplectic form on Q. There is an associated almost complex structure J on Q such that

$$t\,\sigma(x, y) = g(x, Jy),$$

where $t = g(V, V)^{-1/2}$ measures the size of the fibres.

<u>Proposition</u> [AS] If Q is Kähler (i.e. J is integrable), the Hamiltonian flow on Q generated by t preserves the $SU(3)$ structure:

$$\mathcal{L}_{J\mathrm{grad}t}\,\sigma = 0, \quad \mathcal{L}_{J\mathrm{grad}t}\,\Omega = 0.$$

Assume this flow arises from a second S^1 action:

$$\begin{array}{ccc}
H_c & & \\
\big\downarrow S^1 & & \\
t^{-1}(c) & \stackrel{\mathbb{R}^*}{\longleftarrow} & Q' \\
\big\downarrow S^1 & \swarrow \mathbb{C}^* & \\
M^4 & &
\end{array}$$

The 4-manifold

$$M = \frac{t^{-1}(c)}{S^1} \cong \frac{H}{T^2}$$

then has

- a complex structure and holomorphic 2-form $\omega_2 + i\omega_3$,
- a 1-parameter family of Kähler forms $\tilde{\omega}_1 = \tilde{\omega}_1(t)$, such that

If $f:(a,b)\times M\to\mathbb{R}$ is defined by

$$t\,\tilde\omega_1\wedge\tilde\omega_1 = f\,\omega_2\wedge\omega_2$$

so that

$$f = t\,\frac{\text{symplectic volume}}{\text{complex volume}},$$

then

$$\tilde\omega_1''(t) = \boxed{\frac{\partial^2\tilde\omega_1}{\partial t^2} = 2i\partial\overline{\partial}f}$$

Setting $\xi = fJdt$ allows one to recover the closed forms

$$\begin{aligned}
\varphi =\ & \tilde\omega_1\wedge\eta + t\omega_2\wedge\xi + tf\omega_3\wedge dt\ +\ dt\wedge\xi\wedge\eta\\
\in\ & \textstyle\bigwedge^2\mathfrak{base}\otimes\bigwedge^1\mathfrak{fibre}\quad\oplus\quad\bigwedge^3\mathfrak{fibre},\\
\varphi^* =\ & \tfrac{1}{2}t^2\tilde\omega_1\wedge\tilde\omega_1 + t^2\tilde\omega_1\wedge dt\wedge\xi + f\omega_2\wedge\eta\wedge dt + \omega_3\wedge\xi\wedge\eta\\
\in\ & \textstyle\bigwedge^4\mathfrak{base}\quad\oplus\quad\bigwedge^2\mathfrak{fibre}\otimes\bigwedge^2\mathfrak{base},
\end{aligned}$$

corresponding to the inclusion $SO(4)\subset G_2$.

Hyperkähler metrics

A hyperkähler structure on a 4-manifold M is defined by a triple of symplectic forms $\omega_1, \omega_2, \omega_3$ with $\omega_i \wedge \omega_j$ zero if $i \neq j$ and independent of $i = j$. Then $h(x,y) = \omega_1(\omega_2 x, \omega_3 y)$ defines a Riemannian metric with Hol$(h) = SU(2)$ and

$$J_1 = \omega_2^{-1}\omega_3, \quad J_2 = \omega_3^{-1}\omega_1, \quad J_3 = \omega_1^{-1}\omega_1$$

are compatible complex structures.

If f is constant on M, then

$$\tilde{\omega}_1 = (p + qt)\omega_0 + (r + st)\omega_1$$

with $-\omega_0 \wedge \omega_0 = \omega_1 \wedge \omega_1$ and $(r+st)^2 - (p+qt)^2 > 0$. M must have a reverse *almost-Kähler* structure, unless $p = q = 0$.

One may choose an orthonormal basis of 1-forms so that

$\omega_0 = e^{12} - e^{34}$	$\omega_1 = e^{14} + e^{23}$ $\omega_2 = e^{12} + e^{34}$ $\omega_3 = e^{13} + e^{42}$

For instance, setting

$$e^1 = \sqrt{z}\,dx, \quad e^2 = \sqrt{z}\,dy, \quad e^3 = \sqrt{z}\,dz$$
$$e^4 = \frac{1}{\sqrt{z}}(dt - x\,dy)$$

gives closed forms

$$\begin{aligned}
\omega_0 &= z\,dx \wedge dz - dt \wedge dy \\
\omega_1 &= z\,dx \wedge dy + dz \wedge dt + x\,dy \wedge dz \\
\omega_2 &= z\,dx \wedge dz + dt \wedge dy \\
\omega_3 &= dx \wedge dt - x\,dx \wedge dy + z\,dy \wedge dz.
\end{aligned}$$

The resulting Gibbons-Hawking metric

$$z(dx^2 + dy^2 + dz^2) + \frac{1}{z}(dt - x\,dy)^2$$

is conformally equivalent to a left-invariant self-dual metric on a solvable group $\mathbb{C}H^2$.

First example

The construction gives various metrics with G_2 holonomy on manifolds $H^6\times(a,b)$ where H is a T^2 bundle over T^4. For instance, let $H=(G/\Gamma)\times(G/\Gamma)$ where

$$G=\left\{\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix} : a,b,c\in\mathbb{R}\right\}.$$

Then H has a basis of 1-forms (e^i) with

$$de^i=\begin{cases}0 & \text{if} \quad i=1,2,3,4\\ e^{12} & \qquad i=5\\ e^{34} & \qquad i=6,\end{cases}$$

If $\xi = e^5 - e^6$ and $\eta = -e^5 - e^6$ then $d\xi=\omega_0$ and $d\eta=-\omega_2$.

Take $(p,q,r,s)=(0,1,1,0)$, so $f=t(1-t^2)$ with $|t|<1$:

$$\begin{aligned}\varphi &= \left[-2e^{56}+t^2(1-t^2)(e^{13}+e^{42})\right]\wedge dt\\ &\qquad +2t(e^{345}-e^{126})-e^{145}-e^{146}-e^{235}-e^{236}\\ &\overset{?}{=} E^{145}+E^{235}+E^{127}+E^{347}+E^{136}+E^{426}+E^{567}\end{aligned}$$

The associated orthonormal basis is

$$E^1 = \sqrt{t(1-t^2)}e^1, \quad E^2 = \sqrt{t}(e^3 - te^1),$$
$$E^3 = \sqrt{t(1-t^2)}e^4, \quad E^4 = \sqrt{t}(e^2 + te^4),$$
$$E^5 = \frac{1}{1-t^2}\xi, \quad E^6 = \frac{1}{t}\eta, \quad E^7 = t\sqrt{1-t^2}dt.$$

Setting $t = \cos\theta = \cos_\theta$ gives the

<u>Corollary</u> (i) $(G/\Gamma)^2 \times (0, \frac{\pi}{2})$ admits a Ricci-flat metric

$$\begin{aligned} g \;=\; & \cos_\theta \sum_{i=1}^{4} (e^i)^2 + 2\cos_\theta^2(-e^1 \odot e^3 + e^2 \odot e^4) \\ & +(\csc_\theta \xi)^2 + (\sec_\theta \eta)^2 + (\cos_\theta \sin_\theta^2 \, d\theta)^2, \end{aligned}$$

induced from a Kähler metric on $Q = (G/\Gamma) \times T^2 \times (0, \frac{\pi}{2})$.

(ii) The associated 3-form $\boxed{\varphi = \Sigma \wedge (\cos_\theta \sin_\theta^2 d\theta) + \Phi^+}$ is described by forms

$$\begin{aligned} \Phi^+ \;&=\; \text{(a constant 3-form)} - 2\cos_\theta d(e^{56}), \\ \Sigma \wedge \Sigma \;&=\; \text{(a constant 4-form)} + \cos_\theta^2 \sin_\theta^2 e^{1234} \end{aligned}$$

evolving on H along a quartic $y = \frac{1}{2}x^2(1 - \frac{1}{4}x^2)$ in $\mathbb{R}^2$.

Volume forms

Let $(M^4,\omega_1,\omega_2,\omega_3)$ be hyperkähler, and consider the complex structure J_1. If $\tilde\omega_1=\omega_1+i\partial\overline\partial u$ then

$$\tilde\omega_1\wedge\tilde\omega_1=\mathcal{M}(u)\,\omega_1\wedge\omega_1,$$

where $\mathcal{M}$ is the complex Monge-Ampère operator. The Ricci form of $\tilde\omega_1$ is $\tilde\rho=-i\partial\overline\partial\log\mathcal{M}(u)$.

Theorem Given real analytic data u,v on M with $\omega_1{+}i\partial\overline\partial u$ $+$definite, there exists $a>0$ and $U:(0,a)\times M\to\mathbb{R}$ to

$$\tfrac12 U''(t)=t\,\mathcal{M}(U)\quad(=f),\qquad U(0)=u,\ U'(0)=v,$$

with $\omega_1{+}i\partial\overline\partial U$ is $+$definite for any $t\in(0,a)$.

Corollary Working on $M\times\mathbb{R}^2_{(p,q)}\times(0,a)$, set $\xi=dp-\frac12 JdU'$ and assume that $\omega_2=-d\eta$. Then

$$\varphi=\tilde\omega_1\wedge\eta+t\,\omega_2\wedge\xi+tf\omega_3\wedge dt+dt\wedge\xi\wedge\eta$$

defines a Riemannian metric g with $\mathrm{Hol}(g)\subseteq G_2$.

Second example

Consider the case in which $\partial\overline{\partial}U \wedge \partial\overline{\partial}U = 0$, so

$$\mathcal{M}(U) = (\omega_1 + i\partial\overline{\partial}U)^2 = {\omega_1}^2 + 2i\omega_1 \wedge \partial\overline{\partial}U,$$

and $\mathcal{M} = 1 + 2\Delta$.

Given the standard hyperkähler structure on $\mathbb{C}^2$ with

$$\omega_1 = \tfrac{1}{2}i(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2), \quad w_2 + i\omega_3 = dz_1 \wedge dz_2,$$

take $U = U(t, z_1)$ independent of z_2. Setting $z_1 = x + iy$ and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, there are separable solutions

$$U(t, x, y) = \tfrac{1}{3}t^3 + \ell(t)m(x, y),$$

where $\ell(t)$ is a solution of the Airy equation $\ell'' - ct\ell = 0$ and $\Delta m + cm = 0$.

For instance,

$$\begin{aligned}\varphi = (1 + \mathrm{Ai}(t)\sin x)(e^{126} + t^2e^{147} + t^2e^{237})\\ -\mathrm{Ai}'(t)\cos x(te^{132} + e^{267}) + e^{346} + e^{567} + te^{135} + te^{425},\end{aligned}$$

with $-de^6 = e^{13} + e^{42}$ and $e^7 = dt$ on $H^6 \times (0, a)$.