# Geometry & Topology of Wolf Spaces

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## Contents

## 1.1 Riemannian symmetric spaces

$$M^d = \frac{G}{H}$$

If the isometry group $G$ acts faithfully then $H$ is the holonomy group and $H \subset O(d)$.

The action of $H$ on each tangent space $T_m M$ can give a model for more general Riemannian manifolds:

Kähler, quaternion-Kähler, $H$-structure with torsion, ...

Some aspects of the topology only depend on the holonomy $H$. Others depend on $G$; spaces with a common isometry group have a hidden affinity $^\star$

## 1.2 Quaternionic symmetric spaces

are analogues of the Hermitian symmetric spaces. The classical compact ones of real dimension $4n$ are

$$\mathbb{HP}^n = \frac{Sp(n+1)}{Sp(n)\times Sp(1)}$$

$$\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+2}) = \frac{SU(n+2)}{S(U(n)\times U(2))}$$

$$\mathbb{G}\mathrm{r}_4(\mathbb{R}^{n+4}) = \frac{SO(n+4)}{SO(n)\times SO(4)}.$$

Of these, only $\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+2})$ (and $\mathbb{G}\mathrm{r}_4(\mathbb{R}^6)$ ) are Kähler.

Exceptional ones have real dimensions $8, 28, 40, 64, 112$ :

$$\frac{G_2}{SO(4)},\quad \frac{F_4}{Sp(3)Sp(1)},\quad \frac{E_6}{SU(6)Sp(1)},\quad \frac{E_7}{Spin(12)\,Sp(1)},\quad \frac{E_8}{E_7\,Sp(1)}.$$

Recall that $SO(4) = Sp(1)Sp(1) = Sp(1)\times_{\mathbb{Z}_2}Sp(1)$ is not simple.

## 1.3 Wolf's construction

Given a compact simple Lie algebra $\mathfrak{g}$, choose a Lie subalgebra $\mathfrak{su}(2)=\mathfrak{sp}(1)$ arising from a highest root. Set

$$H = KSp(1) = \{g \in G : \mathrm{Ad}(g)(\mathfrak{su}(2)) = \mathfrak{su}(2)\}.$$

Then

$$M = \frac{G}{KSp(1)} = \frac{G}{H}$$

is quaternion-Kähler (QK), meaning

$$\boxed{H \subseteq Sp(n)Sp(1)} \subset SO(4n).$$

This means that $M$ admits a parallel 4-form $\Omega$ equivalent to

$1234 + 5678 + \frac{1}{3}(1256 + 1278 + 3456 + 3478 + 1357 + 1386 + 4257 + 4286 + 1458 + 1467 + 2358 + 2367).$

All compact QK homogeneous spaces arise like this (Alekseevsky). What happens if we take other $\mathfrak{su}(2)$'s in $\mathfrak{g}$?

## 1.4 The isotropy representations

of these spaces have special merit. For each Wolf space $G/K\,Sp(1)$, we get a symplectic representation $K \to \mathrm{End}(\mathbb{C}^{2n})$.

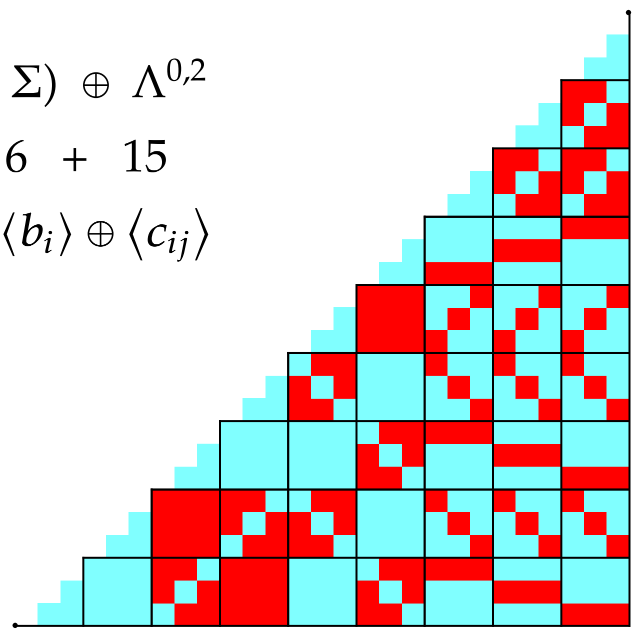
**Example.** Consider $\mathfrak{e}_6 = \mathfrak{su}(6) \oplus \mathfrak{sp}(1) \oplus \mathfrak{m}$, where

$$\mathfrak{m}_c = \Lambda^{3,0} \otimes \Sigma = \mathbb{C}^{40}, \qquad \Sigma = \mathbb{C}^2.$$

But $E_6$ also acts on

$$\begin{aligned} \mathbb{C}^{27} &= (\Lambda^{1,0} \otimes \Sigma) \oplus \Lambda^{0,2} \\ &= \ 6 \ + \ 6 \ + \ 15 \\ &= \langle a_i \rangle \oplus \langle b_i \rangle \oplus \langle c_{ij} \rangle \end{aligned}$$

giving Schläfli's configuration of the 27 lines on a cubic surface:



## 2.1 Nilpotent coadjoint orbits

can obtained [JM] by choosing $\mathfrak{su}(2)\subset\mathfrak{g}$ and setting

$$\mathscr{N}=(\mathrm{Ad}\,G_c)(e)\subset\mathfrak{g}_c,\qquad e=\begin{pmatrix}0&1\\0&0\end{pmatrix}\in\mathfrak{sl}(2,\mathbb{C}).$$

Kronheimer proved that $Z=\mathscr{N}$ admits a hyperkähler metric, but $\mathscr{N}/\mathbb{C}^*$ is compact only if $\mathscr{N}$ is minimal. In this case, $Z=G/KU(1)$ is the so-called twistor space that fibres over $G/K\,Sp(1)$.

**Example.** For $G_2$ there are four non-zero orbits:

$$\begin{array}{llll}\mathfrak{su}(2)_+ & \subset\mathfrak{so}(4) & \subset & \mathfrak{g}_2\\ \mathfrak{su}(2)_- & \subset\mathfrak{so}(4) & \subset & \mathfrak{g}_2\\ \mathfrak{so}(3) & \subset\mathfrak{so}(4) & \subset & \mathfrak{g}_2\\ \mathfrak{so}(3)_{\mathrm{pr}} & & \subset & \mathfrak{g}_2\end{array}$$

Then

$$Z=\frac{G_2}{U(2)_+}\longrightarrow\frac{G_2}{SO(4)}=M^8.$$

By contrast,

$$\mathbb{G}\mathrm{r}_2(\mathbb{R}^7)\cong\frac{G_2}{U(2)_-}\longrightarrow\frac{G_2}{SU(3)}=S^6,$$

in which $SU(3)$ is the fixed point set of an automorphism of order 3 on $G_2$.

## 2.2 Calibrations

The fundamental 3-form

$$F(X,Y,Z) = \langle [X,Y],\, Z\rangle$$

on the Lie algebra $\mathfrak{g}$ defines a function $f$ on $\mathbb{G} = \mathbb{G}\mathrm{r}_3(\mathfrak{g})$ for which

(i) $V \in \mathbb{G}$ is critical iff $V$ is a subalgebra;

(ii) $f$ achieves its maximum on the Wolf space parametrizing minimal $\mathfrak{su}(2)$'s;

(iii) we can easily compute $\mathrm{Hess}(f)$ at any $V = \mathfrak{su}(2)$.

**Example.** Let $V = \mathfrak{so}(3)_{\mathrm{pr}} \subset \mathfrak{su}(3)$. Then $\mathfrak{su}(3)_c \cong \Sigma^2 + \Sigma^4$ where $\Sigma^q = S^q(\mathbb{C}^2)$, and

$$T_V\mathbb{G} \;\cong\; V \otimes V^\perp \;\cong\; \Sigma^2 \otimes \Sigma^4 \;\cong\; \underset{+}{\Sigma^2} \oplus \underset{0}{\Sigma^4} \oplus \underset{-}{\Sigma^6}$$

Whilst the critical manifold $C^5 = \dfrac{SU(3)}{\mathbb{Z}_3 SO(3)}$ has tangent space $\Sigma^4$, both

$$\Sigma^2 \oplus \Sigma^4 \;\cong\; \Sigma^3 \otimes \Sigma^1$$

$$\Sigma^4 \oplus \Sigma^6 \;\cong\; \Sigma^5 \otimes \Sigma^1$$

are quaternionic or $Sp(2)Sp(1)$ modules.

## 2.3 Morse theory

The associated *unstable manifold* $U^8$ is the union of $C^5$ and the upward flow lines of the vector field $\operatorname{grad} f$. It is diffeomorphic to a rank 5 vector bundle over $C^5$ with fibre $\Sigma^4$, and $T_cU = \Sigma^2 \oplus \Sigma^4$. Moreover [G], it is a $\mathbb{Z}_3$ quotient

$$U^8 = \frac{1}{\mathbb{Z}_3}\left(\frac{G_2}{SO(4)} \setminus \mathbb{CP}^2\right).$$

More generally, if $G$ is any compact simple Lie group,

**Theorem** [S]. $f$ is a Morse-Bott function on $\mathbb{G}\mathrm{r}_3(\mathfrak{g})$. The unstable manifold determined by a critical manifold containing $\mathfrak{su}(2) \subset \mathfrak{g}$ is QK and its twistor space is $\mathscr{N}/\mathbb{C}^*$.

A discrete version of the construction (and Nahm's equations) gives rise to the following dynamical system. Given a subspace $V = \langle v_1, v_2, v_3 \rangle \subset \mathfrak{g}$, define

$$V' = \langle\, [v_2, v_3], [v_3, v_1], [v_1, v_2] \,\rangle\,.$$

For generic $V$, one expects

$$V^{(n)} \to \mathfrak{su}(2)_{\min} \in \frac{G}{K\,Sp(1)} \quad \text{as } n \to \infty.$$

## 3.1 The twistor space

The total space of the fibration

$$Z = \frac{G}{KU(1)} \xrightarrow{\pi} \frac{G}{KSp(1)} = M$$

is a real adjoint orbit in $\mathfrak{g}$ and a polarized variety. Wolf pointed out that $Z$ has a complex contact structure $\theta$.

**Example.** $\mathbb{CP}^{2n+1}(\to \mathbb{HP}^n)$ has anticanonical bundle $\overline{\kappa} = \mathcal{O}(2n{+}2)$.

In general, only $L = \mathcal{O}(2)$ is defined and $Z$ is Fano of index $n+1$. There is a holomorphic short exact sequence

$$0 \longrightarrow D \longrightarrow TZ \xrightarrow{\theta} L \longrightarrow 0$$

of vector bundles, in which $D$ is a horizontal distribution and $\theta \in H^0(Z, \Omega^1(L))$. The fibre

$$\pi^{-1}(m) \cong \frac{Sp(1)}{U(1)} = \mathbb{CP}^1 = S^2$$

parametrizes compatible almost complex structures on $T_mM$ and has normal bundle $2n\mathcal{O}(1)$.

## 3.2 The Penrose correspondence

between $M$ and $Z$ is much more general:

| $M$ positive QK | $Z$ contact Fano |
|---|---|
| point | rational curve |
| complex structure | holomorphic section |
| $b_2(M)+1$ | $= b_2(Z)$ |
| Killing field $X$ | $s \in H^0(Z, \mathcal{O}(2))$ |
| Dirac operator | $\overline{\partial}$ on $\Lambda^{0,*} \otimes \mathcal{O}(-n)$ |

The interpretation of solutions to linear field equations as elements of Čech cohomology is the essence of the Penrose programme.

**Big questions.** Is every compact QK manifold ( $H \subseteq Sp(n)Sp(1)$, automatically Einstein) with scalar curvature $s > 0$ necessarily symmetric? Is every contact Fano manifold $Z^{2n+1}$ homogeneous? Open if $n \geqslant 3$.

## 3.3 A moment mapping

Suppose that $M^{4n}$ is a QK manifold with an isometry group $G$ with $\dim G = \ell$. Consider the morphism

$$\begin{array}{rcl}\Phi\colon Z & \to & \mathbb{P}(\mathfrak{g}_c^*) = \mathbb{P}(H^0(Z, \mathcal{O}(L))^*)\\ z & \mapsto & [s_1, \ldots, s_\ell],\end{array}$$

a moment map for the contact structure $\theta$ preserved by $G_c$.

Suppose $\wp \in S^k\mathfrak{g}^*$ is an invariant polynomial. Then either

(a) the image of $\wp^\sharp \in S^k\mathfrak{g}_c \to H^0(Z, \mathcal{O}(L^k))$ is non-zero, or

(b) $\Phi(Z)$ lies in the zero set of $\wp$.

In (a), the image of $\wp^\sharp$ vanishes on $k$ local sections of $Z \to M$ each of which determines a $G$*-invariant* complex structure of type $aI + bJ + cK$. If these are not present, then (b) asserts that $\Phi(Z)$ lies in the nilpotent variety in $\mathbb{P}(\mathfrak{g}_c)$.

**Related question.** Does a positive QK manifold $M^{4n}$ always have isometries?
Yes, at least if $n \leqslant 4$.

## 4.1 Witten rigidity

Let $M^{4n}$ be a Wolf space or QK manifold with isometry group $G$. Its virtual $Spin(4n)$ representation is

$$\Delta_+ - \Delta_- = \Lambda_0^n(E - \Sigma^1) = \bigoplus_{p+q=n} (-1)^p R^{p,q},$$

where $R^{p,q} = \Lambda_0^p E \otimes \Sigma^q$ with $E = \mathbb{C}^{2n}$, $\Sigma^q = S^q(\mathbb{C}^2)$.

The coupled Dirac operator

$$\Gamma(M, \Delta_+ \otimes R^{p,q}) \longrightarrow \Gamma(M, \Delta_- \otimes R^{p,q})$$

has index $i^{p,q} = \int_M \mathrm{ch}(R^{p,q})\hat{A}(M)$.

**Theorem.** $(-1)^p i^{p,q} = \begin{cases} 0 & \text{if } p+q < n, \\ b_{2p-2} + b_{2p} & \text{if } p+q = n, \\ \dim G & p=0,\ q=n+2. \end{cases}$

This is a $G$-equivariant statement, and if $p+q \leqslant n$ the associated $G$-modules are trivial.

## 4.2 Application to dimension 8

Index theory (and the $\gamma$ filtration) gives a linear constraint on the Betti numbers and estimates on the isometry group, in terms of characteristic classes including the integral class $u \in H^4(M,\mathbb{Z})$ that represents $\Omega$.

**Example.** If $d = \dim M = 8$ then

$$b_2 + 1 = b_4.$$

This suggests that $b_2 = 0$ or $1$! Moreover

$$\dim G = 5 + \int_M u^2.$$

If $b_4 = 1$ then

$$\dim G = \begin{cases} 5+16 & = \dim Sp(3), \\ 5+9 & = \dim G_2, \\ 5+4 & = \dim Sp(1)^3, \\ 5+1 & = \dim SO(4), \end{cases}$$

corresponding to

$$\mathbb{HP}^2 = \frac{Sp(3)}{Sp(2)\times Sp(1)}, \qquad \frac{G_2}{SO(4)}, \qquad \frac{\mathbb{HP}^2}{(\mathbb{Z}_2)^2}, \qquad ?$$

Only the first two spaces are non-singular.

## 4.3 Towards a classification

Let $M^{4n}$ be a compact positive QK manifold.

**Theorem** [LS,W]. If $b_2(M) > 0$ then $M$ is isometric to $\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+2})$.

Proof uses Mori theory on the twistor space $Z$. If $b_2(Z) > 1$ there exists a second family of rational curves on $Z$ transverse to the fibres over $M$, and a Fano contraction

$$Z \longrightarrow \mathbb{CP}^{n+1}$$

with *its* fibres tangent to the contact dstribution $D$. This forces $Z = \mathbb{P}(T^*\mathbb{CP}^{n+1})$.

**Corollary** [GMS]. The only Wolf spaces with a (stably) almost complex structure are $\mathbb{HP}^1 = S^4$ and $\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+2})$.

Proof. If $n > 1$ then

$$R^{1,n-3} \oplus R^{1,n-1} \cong (E \otimes \Sigma^1) \otimes \Sigma^{n-2} \cong (T^{1,0} \oplus T^{0,1}) \otimes \Sigma^{n-2},$$

forcing $-i^{1,n-1} = 1 + b_2$ to be even.

## 4.4 Spin and the Â genus

Let $M^{4n}$ be compact, $H \subseteq Sp(n)Sp(1)$ and $s > 0$.

Key fact: if we ignore $\mathbb{HP}^n$ then $M$ is spin iff $n$ is even. In this case, $\hat{A}(M) = 0$ because $s > 0$. There is a dichotomy according to the parity of $n$.

**Theorem** [PS]. A positive QK manifold $M^8$ is isometric to a Wolf space.

Attempts to push this to dimension 12 relied on elliptic genera [HH], but appear to need the assumption $\hat{A}(M) = 0$. Significant progress has been made recently by Amann in higher dimensions:

**Theorem.** If $b_4 = 1$ and $3 \leqslant n \leqslant 6$ then $M \cong \mathbb{HP}^n$.

All exceptional Wolf space have $b_4 = 1$, including $\dfrac{F_4}{Sp(3)Sp(1)}$.

**Theorem** [A]. If $n = 5$ and $\hat{A}(M) = 0$ then $\dim G \geqslant 15$ and $M$ is a Wolf space if (for example) $\int_M u^5 > 384$.

## 5.1 Betti numbers of symmetric spaces

Consider the Poincaré polynomial

$$P(t) = 1 + b_1 t + b_2 t^2 + b_3 t^3 + \cdots$$

and assume Euler characteristic $\chi = P(-1) \neq 0$. Then

$$\log P(t-1) = \log \chi - d\,t + \phi\,t^2 + \cdots$$

where $d = \dim M$, and

$$2\phi = \frac{P''(-1)}{2\,\chi} - \tfrac{1}{8}d^2.$$

By construction, this coefficient is additive for products:

$$\phi(M \times N) = \phi(M) + \phi(N).$$

**Theorems** (i) If $M^{4n}$ is compact hyperkähler, $\chi = 0$ or $\phi = -\frac{5}{6}n$.
$\star$ (ii) [FS]. If $M^d = G/H$ is an irreducible compact SS of type ADE or a HSS,

$$\phi = \tfrac{1}{12}(h(\mathfrak{g}) - 2)d,$$

where $h(\mathfrak{g})$ is the Coxeter number. If $M^{4n}$ is an ADE Wolf space then $\phi = \frac{1}{3}n^2$.

## 5.2 The case of $E_8$

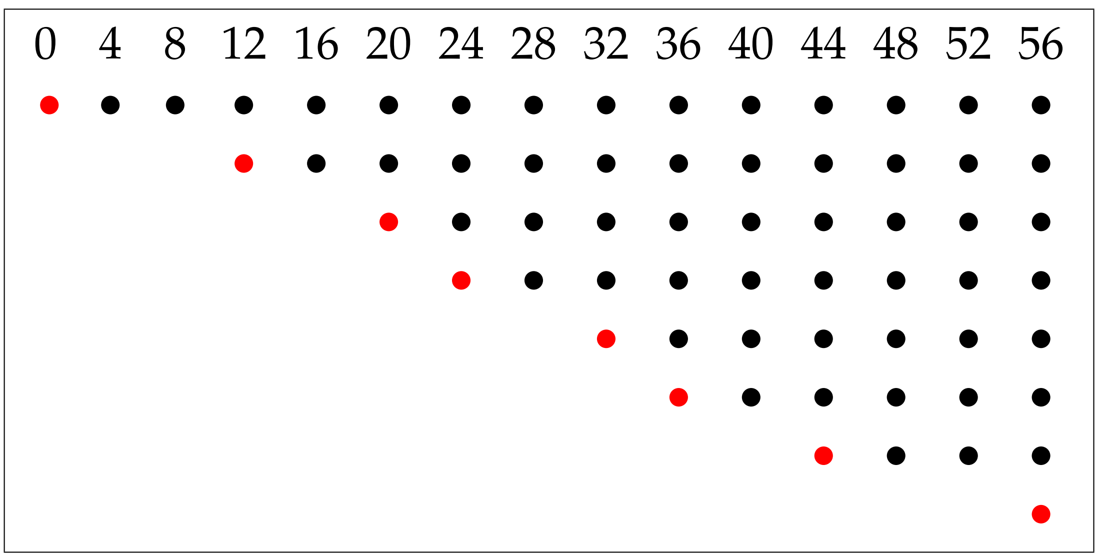
The odd Betti numbers of a positive QK manifold $M^{4n}$ all vanish and the intersection form is definite: $b_{2i+1} = 0$ and $b_{2n} = b_{2n}^+$.

The signature of an ADE Wolf space space equals its rank: $b_{2n} = r$. Its Euler characteristic $\chi$ equals the number of positive roots.

$E_8/E_7Sp(1)$ has 8 primitive cohomology classes $\sigma_k \in H^{4k}(M, \mathbb{R})$ ;

$$H^{56}(M, \mathbb{R}) = \left\langle \sigma_k \cup u^{14-k} : k = 0, 3, 5, 6, 8, 9, 11, 14 \right\rangle,$$

exhibiting 'secondary Poincaré duality' about degree $n = 28$ :




**Question** [HS]. What happens over the integers? Is the quadratic form $H^{56}(M, \mathbb{Z}) \times H^{56}(M, \mathbb{Z}) \to \mathbb{Z}$ diagonalizable or the $E_8$ lattice? The quaternionic volume is

$$\int_M u^{28} = 2^3.3^2.5^2.7.31.37.41.43.47.53 = \frac{5!\,9!\,57!}{19!23!29!} = 63468758442600.$$

## 5.2 References, see arXiv or MathSciNet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[FS] Fino–Salamon

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[W] Wiśniewski