INTRINSIC TORSION VARIETIES

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in honour of Oldřich Kowalski

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Introduction

Riemannian pre-holonomy

Choose a structure group $H \subset SO(n)$

Reductions are parametrized by the homogeneous space $F = \dfrac{SO(n)}{H}$

An H-structure is a section s of the associated bundle $\begin{array}{c} P \\ \downarrow F \\ M^n \end{array}$

Its intrinsic torsion τ is given by $\nabla^{\mathsf{LC}} s : TM \to TF$

Note that $\mathfrak{so}(n) = \mathfrak{h} \oplus \mathfrak{h}^\perp$ with $\mathfrak{h}^\perp \cong T_{s(m)}F \quad \Rightarrow \quad \tau_m \in \mathbb{R}^n \otimes \mathfrak{h}^\perp$

This measures the "holonomy failure" of s at each point $m \in M$

Examples in six dimensions

. . . arise from choosing $H \subset SO(6)$

Two irreducible non-symmetric holonomies:

	pre-Kähler	pre-Calabi-Yau
H	$U(3)$	$SU(3)$
F	$\frac{SO(6)}{U(3)} = \mathbb{CP}^3$	$\frac{SO(6)}{SU(3)} = \mathbb{RP}^7$

Their reductions can be described using the action of $Spin(6) \cong SU(4)$ on the spaces $\mathbb{C}^4$ and $\mathbb{R}^8$ of spinors

We shall focus on $\mathbb{CP}^3$ and other adjoint orbits in $\mathfrak{so}(6) = \mathfrak{su}(4)$ such as

$$\mathbb{G}\mathrm{r}_2(\mathbb{R}^6) = \frac{SO(6)}{SO(2) \times SO(4)} \cong \frac{U(4)}{U(2) \times U(2)} = \mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$$

Coadjoint orbits

Complex flag manifolds

$$\frac{U(4)}{U(1)\times U(1)\times U(1)\times U(1)}$$

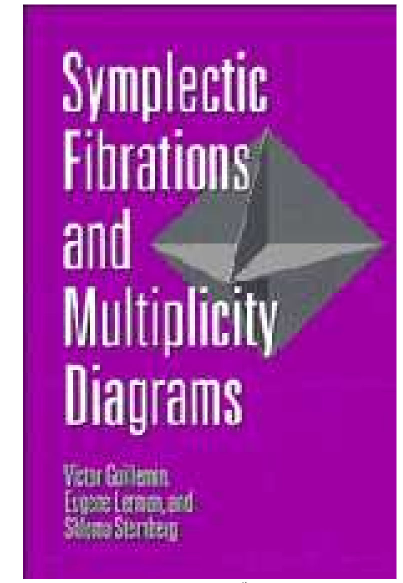
$$\downarrow \; \mathbb{CP}^1$$

$$\frac{U(4)}{U(2)\times U(1)\times U(1)}$$

$$\mathbb{CP}^2 \swarrow \qquad\qquad \searrow \mathbb{CP}^1$$

$$\frac{U(4)}{U(3)\times U(1)} \qquad\qquad \frac{U(4)}{U(2)\times U(2)}$$

Symplectic fibrations



$$\frac{U(4)}{U(1)\times U(1)\times U(1)\times U(1)}$$

$$\downarrow \ \mathbb{CP}^1$$

$$\frac{U(4)}{U(2)\times U(1)\times U(1)}$$

$$\mathbb{CP}^2 \swarrow \qquad\qquad \searrow \mathbb{CP}^1$$

$$\mathbb{CP}^3 \qquad\qquad\qquad \mathbb{G}\mathrm{r}(\mathbb{C}^4)$$

Such fibrations are characterized by the existence of a 2-form ω on the total space for which $d\omega(V_1, V_2, \cdot) = 0$ with V_i vertical. Here we can choose $d\omega = 0$

Borel-Weil theory and practice

Theorem: Let V be an irreducible complex representation of G with highest weight $\lambda \in \mathfrak{t}^*$ and vector $v_\lambda \in V$. Then

$$\mathbb{P}(V) \supset G_c \cdot [v_\lambda] \cong \frac{G_c}{P} = \frac{G}{H} = F$$

and $V \cong H^0(F, \mathcal{O}(L_\lambda))$

Example: $\mathbb{CP}^5 = \mathbb{P}(\Lambda^2\mathbb{C}^4) \supset Q = \dfrac{SL(4,\mathbb{C})}{P} = \mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$

Penrose studied the Klein quadric with $SU(2,2)$ in place of $SU(4)$ and resulting field theory. Baston–Eastwood generalized this to arbitrary flag manifolds

3-symmetric spaces and harmonic maps

Twistor theory used double fibrations to construct new harmonic maps from old (often holomorphic) ones.

The general scheme takes an isotropic map as input:

$$\begin{array}{ccccc} & & \mathscr{F} & & \\ & & \mathbb{CP}^2 \downarrow & \searrow & \\ X & \longrightarrow & \mathbb{CP}^3 & & \mathbb{G}\mathrm{r}_2(\mathbb{C}^4) \end{array}$$

Analogous liftings occur in Musso's CMC set-up:

$$\begin{array}{ccc} & & SL(2,\mathbb{C}) \\ & & \downarrow SU(2) \\ X & \longrightarrow & \mathcal{H}^3 \end{array}$$

Geometrical structures
on 6-manifolds

$SO(6)$ in place of $SU(4)$

$$\frac{SO(6)}{T^3}$$

$$\downarrow \ \mathbb{CP}^1$$

$$\frac{SO(6)}{SO(2)\times U(2)}$$

$$\mathbb{CP}^2 \swarrow \qquad\qquad \searrow \mathbb{CP}^1$$

$$\frac{U(4)}{U(3)\times U(1)} \qquad\qquad \frac{U(4)}{U(2)\times U(2)}$$

The action of T^3 will provide a moment map of each space to $\mathbb{R}^3$

Classification of 2-forms

$$e^{12}+2e^{34}+3e^{56}\in\frac{SO(6)}{T^3}$$

$$\downarrow\ \mathbb{CP}^1$$

$$e^{12}+e^{34}\in\frac{SO(6)}{SO(2)\times U(2)}$$

$$\mathbb{CP}^2\ \swarrow \qquad\qquad \searrow\ \mathbb{CP}^1$$

$$e^{12}+e^{34}+e^{56}\in\frac{SO(6)}{U(3)} \qquad\qquad \frac{SO(6)}{SO(2)\times SO(4)}\ni e^{12}$$

Each space is embedded in $\Lambda^2(\mathbb{R}^6)^*$ as an $SO(6)$-orbit of 2-forms

For example, $\sigma\in SO(6)\cdot e^{12}$ iff $\|\sigma\|=1$ and $\sigma\wedge\sigma=0$

Almost complex and product structures

A point of $\mathscr{F}$ determines an
OCS J and a J-invariant OPS:

$$\mathscr{F} = \frac{SO(6)}{SO(2)\times U(2)}$$

$\mathbb{CP}^2 \swarrow$ $\qquad\qquad$ $\searrow \mathbb{CP}^1$

$\mathbb{CP}^3$

parametrizes OCS's
J with $J^2 = -1$

$\mathbb{G}\mathrm{r}_2(\mathbb{R}^4)$

parametrizes OPS's
$\mathbb{R}^6 = \mathscr{V} \oplus \mathscr{H}$ with
$\dim \mathscr{V} = 2$, $\dim \mathscr{H} = 4$

The Klein correspondence

Recall:

(i) $Q = \mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$ parametrizes $\mathbb{CP}^1$'s in $\mathbb{CP}^3$

(ii) A point $x \in \mathbb{CP}^3$ determines a plane $\Pi_\alpha \cong \mathbb{CP}^2$ in Q

(iii) A point $y \in (\mathbb{CP}^3)^*$ determines a plane $\Pi_\beta \cong \mathbb{CP}^2$ in Q

Interpretation:

(i) Given $\mathbb{R}^6 = \mathscr{V} \oplus \mathscr{H}$ there is a $\mathbb{CP}^1$ of compatible OCS's (parametrized by $\omega \in S^2 \subset \Lambda^2_+ \mathscr{H}$)

(ii) Given an OCS J, we have the J-invariant 2-planes $\langle v, Jv \rangle$

(iii) Given J, we have the oppositely-oriented 2-planes $\langle v, -Jv \rangle$

Mixed structures

A point of $\mathscr{F}$ determines an
OCS J and a J-invariant OPS:

$$\mathscr{F} = \frac{SO(6)}{SO(2)\times U(2)}$$

Let us call an $SO(2)\times U(2)$ structure on a 6-manifold M a ‘mixed structure’. Given the Riemannian metric, it is determined by a section of the associated bundle with fibre $\mathscr{F}$, or by a rank 4 distribution $\mathscr{H} = [\![\mathscr{H}^{1,0}]\!]$ equipped with an almost complex structure

Is this concept worthwhile? What are the key examples and properties?

Double integrability

Definition: Let us say that such a mixed structure on a 6-manifold N is doubly integrable if $\mathrm{Nij}(J) \equiv 0$ and $[\mathscr{V}, \mathscr{V}] \subseteq \mathscr{V}$ ($\mathscr{V}$ being the 2-dimensional distribution)

The local model is then a fibration $\begin{array}{c} N \\ \downarrow \pi \\ M^4 \end{array}$ whose total space N is complex

This captures two very different classes of examples:

(i) N is a holomorphic bundle over a complex surface, and elliptic fibrations (fibre T^2) of importance in deformation theory

(ii) M^4 has a conformal structure with $\mathrm{Weyl}_+ \equiv 0$ and $N \subset \Lambda^2_+ T^*M$ is its twistor space. Each S^2 fibre is a rational curve with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$

Specific examples

(i) N is the Iwasawa manifold $\dfrac{H_{\mathbb{C}}}{\Gamma}$ over $M = T^4$.

Stability theorem: Any invariant complex structure $\mathbb{J}$ on N arises from one, say J, on T^4 and the induced j on the T^2 fibre is determined by J

This and similar examples typically possess **bi**hermitian structures

(ii)

$$\begin{array}{ccc} \mathbb{CP}^3 = \dfrac{SO(5)}{U(2)} & \qquad & \mathbb{F} = \dfrac{SU(3)}{T^2} \\ \downarrow & & \downarrow \\ S^4 = \dfrac{SO(5)}{SO(4)} & & \mathbb{CP}^2 \end{array}$$

Hitchin's theorem: Any Kähler twistor space is either $\mathbb{CP}^3$ or $\mathbb{F}$

Intrinsic torsion

Almost product classes

An $SO(2)\times SO(4)$ structure on M^6 gives $T_mM = \mathscr{V}\oplus\mathscr{H}$

Its intrinsic torsion τ_m lies in

$$\begin{aligned}\mathbb{R}^6\otimes\mathfrak{h}^\perp &\cong (\mathscr{V}\oplus\mathscr{H})\mathscr{V}\mathscr{H}\\ &\cong \mathscr{V}\mathscr{V}\mathscr{H}\oplus\mathscr{H}\mathscr{H}\mathscr{V}\\ &\cong \mathscr{V}\oplus 2\mathscr{H}\oplus\mathscr{H}S^2_0\mathscr{V}\oplus\mathscr{V}S^2_0\mathscr{H}\oplus\mathscr{V}\Lambda^2_+\mathscr{H}\oplus\mathscr{V}\Lambda^2_-\mathscr{H}\end{aligned}$$

and has 7 irreducible components, giving rise to 2^7 classes

This compares with 6 for $O(p)\times O(q)$ giving Naveira's 36 classes

Other Gray-Hervella type classes

$\dim_{\mathbb{R}} F$	TM	# τ-irreducibles
6	$[\![\Lambda^{1,0}]\!]$	4
8	$[\![\mathscr{V}^{1,0}]\!] \oplus \mathscr{H}$	7
10	$[\![\mathscr{V}^{1,0}]\!] \oplus [\![\mathscr{H}^{1,0}]\!]$	16
12	$[\![\mathscr{V}_1^{1,0}]\!] \oplus [\![\mathscr{V}_2^{1,0}]\!] \oplus [\![\mathscr{V}_3^{1,0}]\!]$	36

Good news: The intrinsic torsion of a mixed structure is completed determined by that of the associated OCS and OPS   Proof: $T\mathscr{F} = T\mathbb{CP}^3 + T\mathbb{G}\mathrm{r}_2(\mathbb{R}^6)$

Corollary: The standard mixed structure on the Iwasawa manifold belongs to $\mathscr{W}_3 \cap \mathscr{V}\Lambda^2_+\mathscr{H}$, a single irreducible component of real dimension 2

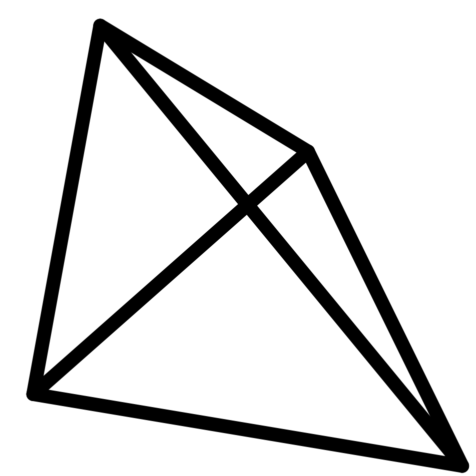
Toric invariance

The moment mapping

...for the action of T^3 on complex projective space has the form

$$\mu\colon \mathbb{CP}^3 \longrightarrow \mathfrak{so}(6)^* = \Lambda^2(\mathbb{R}^6)^* \xrightarrow{\pi} \left\langle e^{12}, e^{34}, e^{56} \right\rangle$$

Its image is a solid tetrahedron $\mathscr{T}$, useful for describing the torsion of nilmanifolds



The maximal torus T^3 acts transitively on $\mu^{-1}(t) \cong T^k$ with $k = 0, 1, 2, 3$, so $\mathbb{CP}^3/T^3 \cong \mathscr{T}$. Vertices, edges and faces correspond to $k = 0, 1, 2$ respectively

Application to nil torsion

Let $(e^1,\dots,e^6)$ be an ON basis of 1-forms on the Iwasawa manifold N^6

According to the fundamental theorem of Riemannian geometry, $\tau = \nabla^{\mathsf{LC}} s$ is completely determined by

$$\begin{aligned}\sum_{r=1}^{6} e^r \otimes de^r &= e^5 \otimes (e^{13} + e^{42}) + e^6 \otimes (e^{14} + e^{23}) \\ &= \mathfrak{Re}\left[(e^5 - ie^6) \otimes (e^1 + ie^2) \wedge (e^3 + ie^4)\right]\end{aligned}$$

and is the real part of an eigenvector for T^3 with weight $-\theta_1 + \theta_2 + \theta_3$

Corollary: Each of the 16 Gray-Hervella subsets of $\mathbb{CP}^3$ is T^3 invariant and determined by its image in $\mathscr{T}$

The moment mapping

. . . for the action of T^3 on the Grassmannian is

$$\mu\colon \mathbb{G}\mathrm{r}_2(\mathbb{R}^6) \xrightarrow{\text{Plücker}} \Lambda^2(\mathbb{R}^6)^* \xrightarrow{\pi} \left\langle e^{12}, e^{34}, e^{56} \right\rangle$$

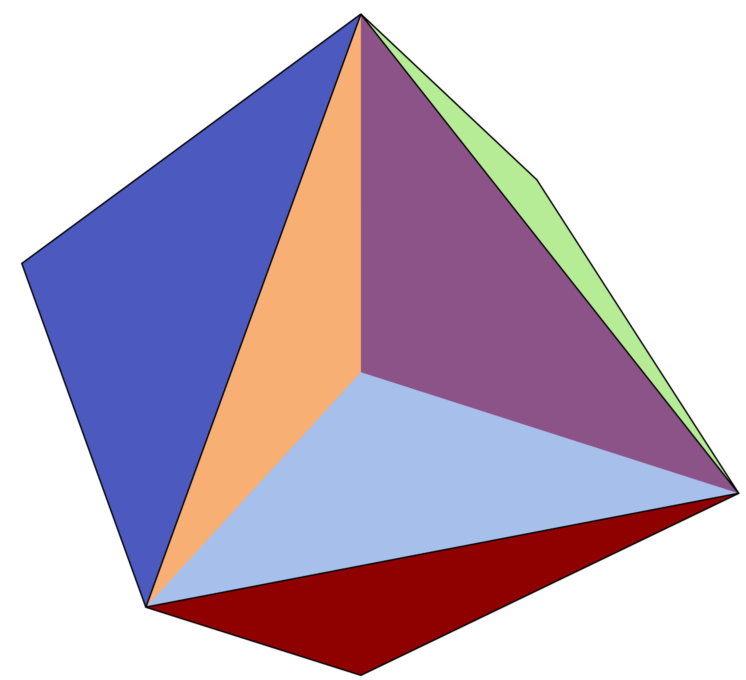
If $\mathscr{V} = \langle \alpha, \beta \rangle$ with $\alpha = \sum a_r e^r$, $\beta = \sum b_r e^r$ orthonormal then

$$\mu(\mathscr{V}) = \pi(\alpha \wedge \beta) = (x, y, z) \quad \text{where} \quad \begin{cases} x = a_1 b_2 - a_2 b_1 \\ y = a_3 b_4 - a_4 b_3 \\ z = a_5 b_6 - a_6 b_5 \end{cases}$$

The Cauchy-Schwartz inequality yields

Proposition: The image of μ satisfies $|x| + |y| + |z| \leqslant 1$, with equality iff $\beta = J\alpha$ where $J = \pm e^{12} \pm e^{34} \pm e^{56}$

The image is a solid octahedron $\mathscr{O}$



The external faces represent $\mathbb{CP}^2$'s that parametrize J-invariant 2-planes. The symplectic quotient $\dfrac{\mu^{-1}(x,y,z)}{T^3}$ is generically $\mathbb{CP}^1$, so $\mathbb{G}\mathrm{r}_2(\mathbb{R}^6)/T^3$ is formed from an S^2 bundle over $\mathscr{O}$ with degenerations on faces, edges and vertices

Concluding example

Let N^6 again be the Iwasawa manifold with its standard metric. Each point $p \in \mathbb{G}\mathrm{r}_2(\mathbb{R}^6) = \mathbb{G}\mathrm{r}_2(\mathfrak{n})$ defines an OPS structure on N via $\mathfrak{n} = \mathscr{V} \oplus \mathscr{H}$ with $\dim \mathscr{H} = 4$, and we may compute its intrinsic torsion

Proposition: If $[\mathscr{H}, \mathscr{H}] \subseteq \mathscr{H}$ then $\mu(p) = (x, y, 0)$ lies in the internal (light blue) coordinate plane in $\mathscr{O}$ representing singular values of μ

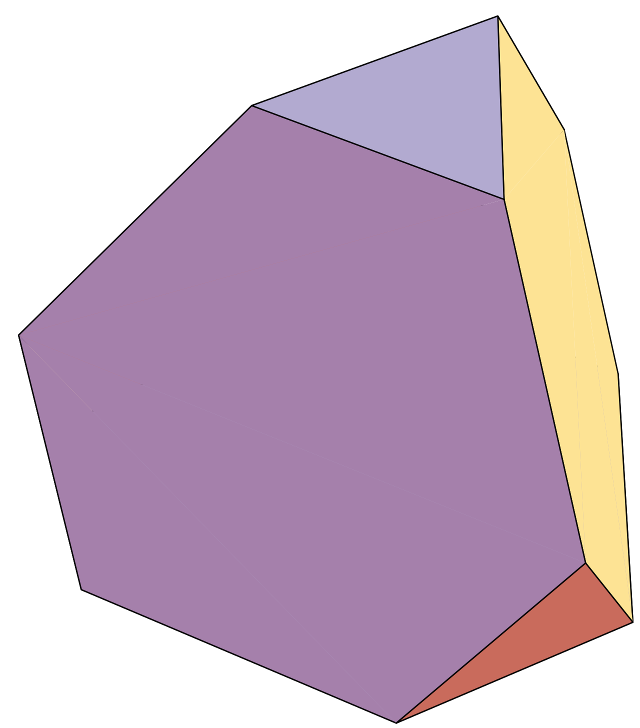
Vanishing of a selection of the 7 torsion classes will be characterized by a subset of $\mathbb{G}\mathrm{r}_2(\mathbb{R}^6)/T^3$. Complementary choices must yield non-intersecting subsets, since the holonomy of N is not $SO(2) \times SO(4)$ and no point has $\tau = 0$.

The images of each subset in the octahredon $\mathscr{O}$ will provide some measure of the associated torsion constraint

References

For a survey of almost-Kähler geometry:
V. Apostolov, T. Drăghici: math.DG/0302152

For the motivation behind this lecture:
V. Guillemin, E. Lerman, S. Sternberg: CUP 1996



Visualizing $\mu\colon \mathscr{F} \to \mathbb{R}^3$