Tensors and representations

Let (M, g) be a Riemannian manifold of dimension d.
Let ϕ be a tensor with $\{a \in SO(n) : a \cdot \phi = \phi\} = G$.

The holonomy group is a subgroup of G iff $\nabla\phi \equiv 0$. Given

$$\bigwedge^2 T^* \cong \mathfrak{so}(d) = \mathfrak{g} \oplus \mathfrak{g}^\perp,$$

<u>Lemma</u> $\nabla\phi$ can be identified with an element of the space

$$T^* \otimes \mathfrak{g}^\perp =: \mathcal{W} = \bigoplus_{i=1}^{N} \mathcal{W}_i,$$

with say N irreducible components.

<u>Examples</u>

d	ϕ	G	N
$2n$	almost complex structure J	$U(n)$	4
$2n$	non-degenerate 2-form ω	$U(n)$	4
7	positive generic 3-form	G_2	4
$4k$	quaternionic 4-form $\sum_1^3 \omega^i \wedge \omega^i$	$Sp(k)Sp(1)$	6

Sixteen classes of almost Hermitian manifolds

Given (M^{2n}, g) with $\phi = J$ and $G = U(n)$,

Proposition [GH 80]

$$\nabla J \in \underbrace{\mathcal{W}_1 \oplus \mathcal{W}_2} \oplus \underbrace{\mathcal{W}_3 \oplus \mathcal{W}_4}$$

Two halves of equal dimension $n^2(n-1)$ lead to a sort of duality:

- $\nabla J \in \mathcal{W}_1 \oplus \mathcal{W}_2 \Leftrightarrow (d\omega)^{1,2} = 0$
- $\nabla J \in \mathcal{W}_3 \oplus \mathcal{W}_4 \Leftrightarrow M$ is Hermitian

M is *Kähler* iff $\nabla J \equiv 0$.

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If M is locally conformally Kähler then $\nabla J \in \mathcal{W}_4$.

Proposition [G 65] If M has $(d\omega)^{1,2} = 0$, and $M' \subset M$ is pseudo-holomorphic, then M' is minimal.

Definition M is *nearly-Kähler* if $\nabla J \in \mathcal{W}_1$, equivalently $(\nabla_X J)X = 0$ for all X.

Basic model is $S^6 = \frac{G_2}{SU(3)}$, but there is a large class of homogeneous examples.

3-symmetric spaces

$M = G/H$ is a 3-symmetric space if H is the fixed point set of an automorphism θ of G with $\theta^3 = 1$. Defining $J = \frac{1}{\sqrt{3}}(2\theta + 1)$ gives a canonical a.c.s. on $T_m M$.

<u>Theorem</u> [WG 68] Any 3-symmetric space has a nearly-Kähler metric.

Classification includes

- generalizations of S^6 with irreducible isotropy (e.g. $\frac{E_8}{SU(9)}$)
- $G \times G = \dfrac{G \times G \times G}{G}$ (e.g. $S^3 \times S^3$)
- twistor spaces over symmetric spaces (e.g. $\mathbb{CP}^3$, $\mathbb{F}^3$)

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The latter are extensively used in the study of minimal surfaces and harmonic maps. Any such Z has an integrable complex structure J_1 in addition to $J = J_2$

Proposition If $f\colon \Sigma \to (Z, J_2)$ is a pseudo-holomorphic curve then $\pi \circ f$ is harmonic. E.g.

$$\begin{tikzcd}
 & \frac{U(p+q+1)}{U(p) \times U(q) \times U(1)} = Z \arrow[d, "\pi"] \\
S^2 = \mathbb{CP}^1 \arrow[ur] \arrow[r] & \frac{U(n+1)}{U(n) \times U(1)} = \mathbb{CP}^n
\end{tikzcd}$$

Metrics with exceptional holonomy

<u>Theorem</u> [G 76] If M^6 is nearly-Kähler (and $\nabla J \neq 0$), it is Einstein.

E.g. $(\mathbb{CP}^3, J_2)$. In fact, $R = sR_{S^6} + R_{CY}$ with scalar curvature $s > 0$.
The theory of Killing spinors implies that the cone $M \times \mathbb{R}^+$ has a Ricci-flat metric with holonomy in G_2.

More generally, if X^7 has a 3-form ϕ defining a G_2-structure, there is a vector cross product $\bigwedge^2 T^* \to T^*$ [G 67], and

$$\nabla\phi \in T^* \otimes \mathfrak{g}_2^\perp \;\cong\; T^* \otimes T^* \;\cong\; \underbrace{\mathbb{R} \oplus S^2_0 T^* \oplus \mathrlap{\underbrace{\phantom{T^* \oplus \mathfrak{g}_2}}_{d{*}\phi}} T^*}_{d\phi} \oplus \mathfrak{g}_2$$

<u>Corollary</u> [FG 82]

- X has holonomy in G_2 iff $d(*\phi) = 0$ and $d\phi = 0$
- $\nabla\phi \in \mathbb{R}$ iff $d\phi = c(*\phi)$ $(\Rightarrow d(*\phi) = 0)$

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In the second case, X^7 has *weak holonomy* G_2 (associative subspaces are preserved by parallel transport [G 71]), and the cone $X \times \mathbb{R}^+$ has a metric with holonomy in $Spin\,7$.
Standard model $S^7 = \dfrac{Spin\,7}{G_2}$ yields the flat metric on $\mathbb{R}^8$, but $\dfrac{SO(5)}{SO(3)} \times \mathbb{R}^+$ has holonomy *equal* to $Spin\,7$.

Curvature theorems

Theorem [G 77] A compact Kähler manifold with nonnegative sectional curvature and s constant is locally symmetric (i.e. $\nabla R \equiv 0$).

On an almost-Hermitian manifold, $R = K + K^\perp$, where

- K satisfies $K(W,X,Y,Z) = K(W,X,JY,JZ)$
- $K^\perp = C(\nabla\nabla J)$ has zero holomorphic sectional curvature

$K, K^\perp$ decompose further under $GL(n,\mathbb{C})$ and $U(n)$.

Proposition [G 76] If M is Hermitian then

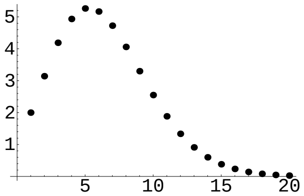
$$\begin{aligned} R(W,X,Y,Z)+R(JW,JX,JY,JZ) &= R(JW,JX,Y,Z)+R(JW,X,JY,Z)+R(JW,X,Y,JZ) \\ &\quad +R(W,JX,JY,Z)+R(W,JX,Y,JZ)+R(W,X,JY,JZ). \end{aligned}$$

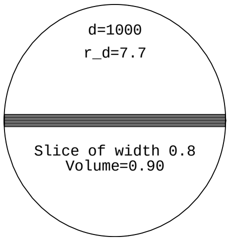
This imposes $k = \frac{1}{6}n^2(n^2-1)$ equations on the Weyl tensor, itself of dimension $< 8k$.

Problem A Riemannian manifold M^{2n} has a *finite* number k of orthogonal complex structures locally. What is the maximum value of k?

Volume $V(r)$ of a small geodesic ball $\mathbb{B}(r)$

- $V(1) = \pi^{d/2}/(d/2)!$ in $\mathbb{R}^d$
- If $V(r_d) = 1$ then $r_d \sim \sqrt{\frac{d}{2\pi e}}$ as $d \to \infty$
- $\mathbb{B}(r_d) \cap (\mathbb{R}^{d-1} \times [-0.4, 0.4])$ has volume $> \ 0.8$





[Zoom in]

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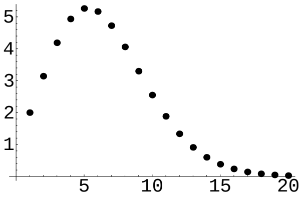

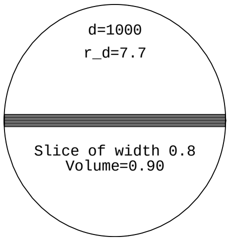

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On a Riemannian manifold,

$$V(r) = \frac{(\pi r^2)^{d/2}}{(d/2)!} \left(1 - \frac{s}{6(d+2)} r^2 + c_4 r^4 + c_6 r^6 + c_8 r^8 + \cdots \right)$$

Theorem [G 73] $c_4 = \dfrac{8\|\mathrm{Ric}\|^2 - 3\|R\|^2 + 5s^2 - 18\Delta s}{360(d+2)(d+4)}$.

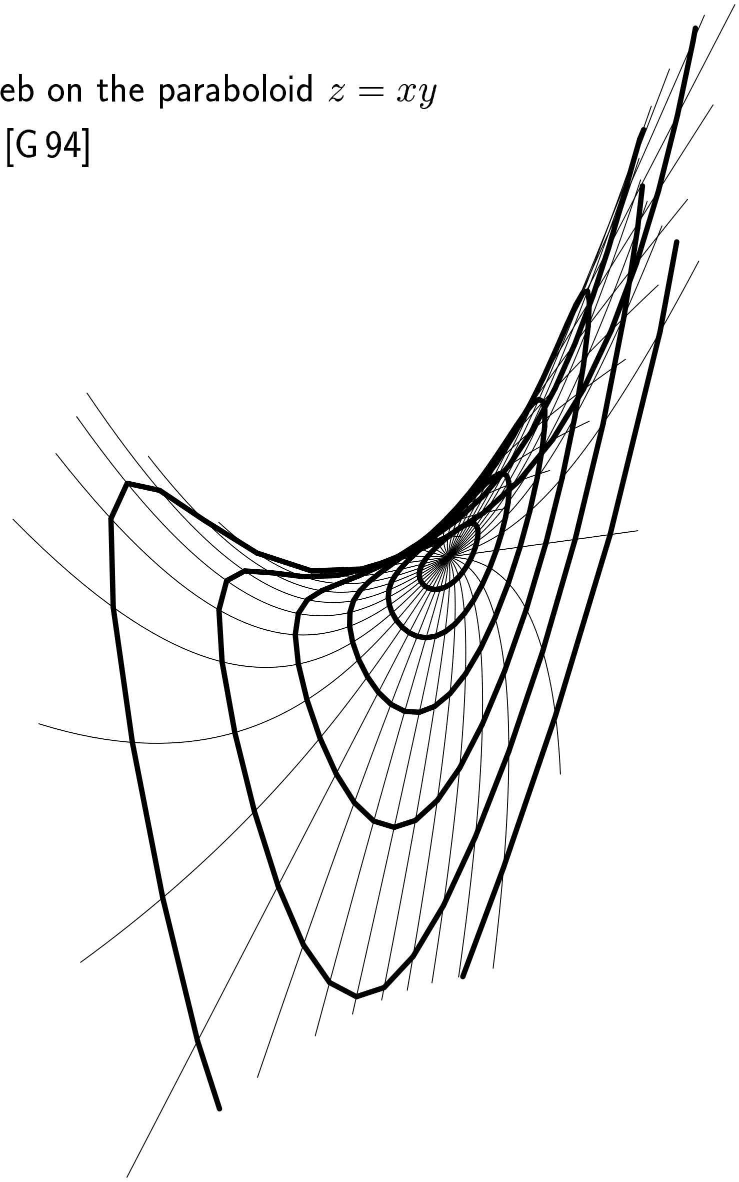
Examples [GV 79] There exist metrics with

- $0 = s = c_4$ and $d = 4$
- $0 = s = c_4 = c_6$ and $d = 734$

Many other asymptotic expansions, and generalizations to tubes.

Theorem [G 88] A tube of radius r surrounding a hypersurface of degree k in $\mathbb{CP}^n$ has volume $\pi^n(1 - (1 - k\sin^2 r)^n)/n!$

Example Geodesic cobweb on the paraboloid $z = xy$
Plotted using `GEOEQ.m` [G 94]



Invariant structures on Lie groups

Any compact simple Lie group G^{2n} admits a complex structure, but no symplectic one. A nilpotent Lie group N^{2n} may or may not admit left-invariant complex or symplectic structures.

Compact nilmanifolds N/Γ never admit Kähler metrics (unless N is abelian). Left-invariant forms provide a minimal model for de Rham cohomology with non-zero Massey products. Dolbeault cohomology is less readily computed:

Proposition [CFG 91] There exist complex nilmanifolds with

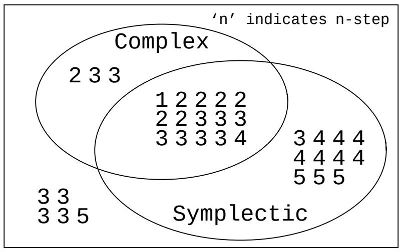
- $n = 4$ and $E_2 \neq E_\infty$
- $n = 6$ and $E_3 \neq E_\infty$.

Theorem [FGM 91] A compact surface (U, ω) of genus $\geqslant 1$ with a symplectomorphism $\varphi: U \to U$ fixing $\mathbf{b} \in H^1(U, \mathbb{Z})$ defines a circle bundle $E \to (U \times [0,1])/\varphi$ that is symplectic and generally non-Kähler.

E.g. For $g = 1$, E is a Kodaira surface with $b_1 = 3$.
A generalization of the construction accounts for all symplectic manifolds with a free S^1 action.

6-dimensional nilmanifolds

<u>Theorem</u> There are 34 isomorphism classes of real 6-dimensional nilpotent Lie algebras $\mathfrak{n}$ admitting structures as shown:

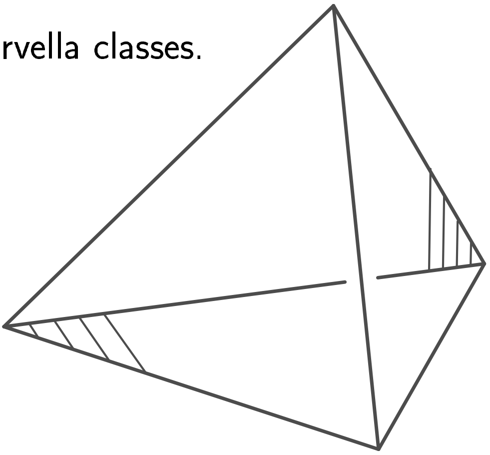


For a metric on $\mathfrak{n}$, almost-Hermitian structures define points of $\dfrac{SO(6)}{U(3)} \cong \mathbb{CP}^3$.

<u>Example</u> For the complex Heisenberg group N or Iwasawa manifold N/Γ,

$$\nabla J \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \Leftrightarrow J \in \mathbb{CP}^2 \cup \mathbb{CP}^2,$$

and these two ‘faces’ contain all 15 proper Gray-Hervella classes.



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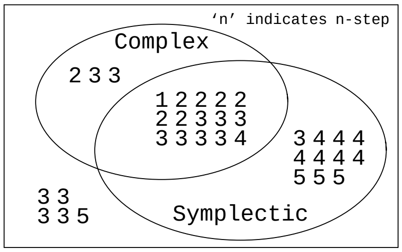

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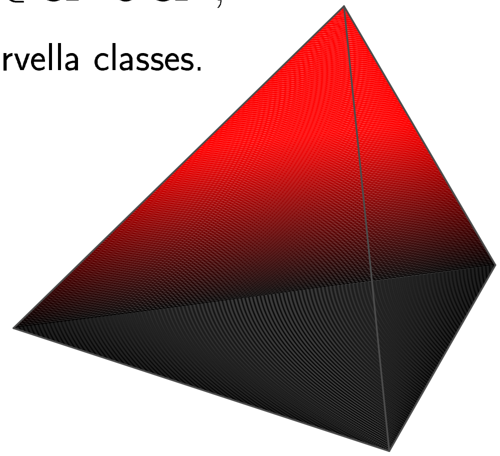

Norm of component of ∇J in each $\mathcal{W}_i$ is represented by respective colour.

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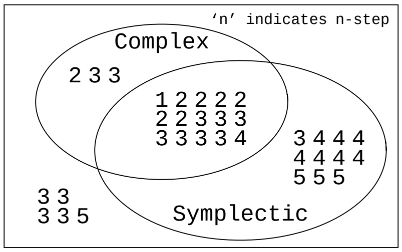

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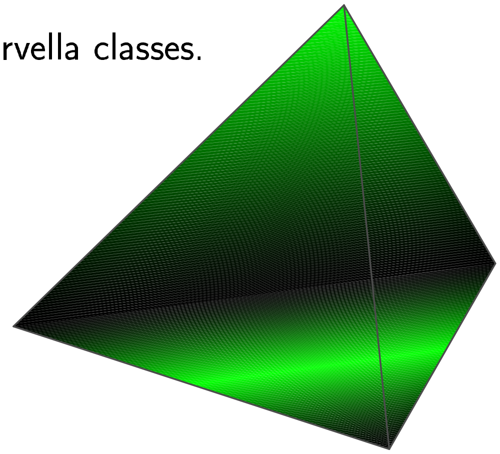
- $\nabla J \in \mathcal{W}_2 \ \Leftrightarrow J \in S^3$



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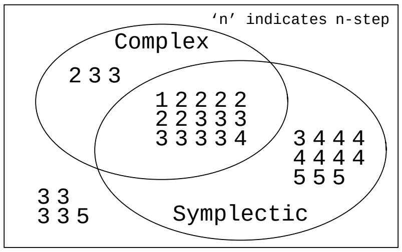

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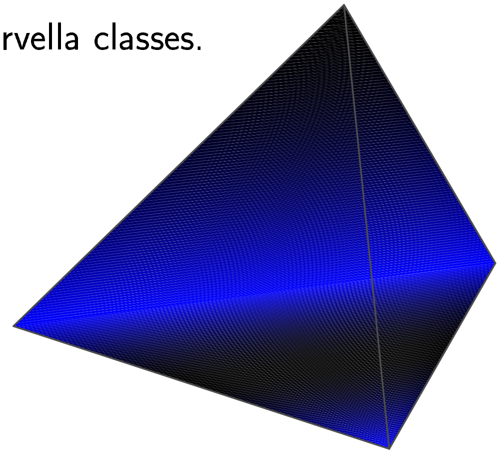

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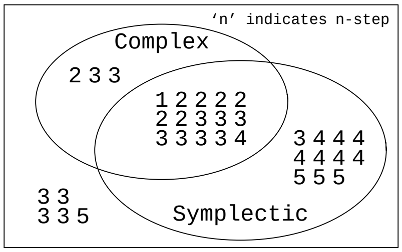

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- $\nabla J \in \mathcal{W}_2 \Leftrightarrow J \in S^3$
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Final picture displays ∇J as a function of position on the two faces. Pure blue indicates Hermitian structures, and green symplectic ones.