Orthogonal Complex Structures

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1. Introduction

Let M be a smooth manifold of dimension $2n$, and let g be a Riemannian metric on M.

An almost-complex structure (abbreviated acs) J on M is an endomorphism of the tangent bundle TM, or equivalently the cotangent bundle T^*M, of M such that $J^2=-1$. Such a tensor induces an orientation on M by taking the $2n$-form $e_1\wedge Je_1\wedge\cdots\wedge e_n\wedge Je_n$ to always be a positive multiple of the volume form. The triple (M,g,J) is called *almost-Hermitian* if J is an orthogonal transformation relative to g, i.e. if

$$g(JX,JY)=g(X,Y)$$

for all tangent vectors X,Y. This equation implies that the tensor ω defined by $\omega(X,Y)=g(JX,Y)$ is anti-symmetric; it is called the fundamental 2-form of the almost-Hermitian structure. Any two of the tensors g,J,ω determine the third.

The acs J is said to be *integrable* if the Nijenhuis tensor

$$N_J(X,Y)=[JX,JY]-J[JX,Y]-J[X,JY]-[X,Y] \tag{1}$$

vanishes. Indeed, the Newlander-Nirenberg theorem [36] implies that $N_J=0$ if and only if (M,J) is a complex manifold in the sense that there exist local complex coordinates $z^1,\dots,z^n$ such that $Jdz^k=i\,dz^k$, $1\le k\le n$. In these circumstances one also says that (M,g,J) is a *Hermitian manifold*.

We shall be concerned with the problem of finding different Hermitian structures on a given Riemannian manifold (M,g), and the following terminology will be convenient.

Definition. An orthogonal complex structure (OCS) on (M,g) is an integrable acs J on M such that $g(JX,JY)=g(X,Y)$.

If M is already oriented then an OCS J may or may not induce the chosen orientation, and according to case we say that J is positively or negatively oriented.

The purpose of this note is to investigate the following

Problem. Given a Riemannian manifold (M,g), does there exist an OCS? If so, describe the set of all OCS's.

The question can be asked either (i) globally, or (ii) locally, and the corresponding questions can be rather different in nature. Given an OCS J defined over a compact manifold, one may ask to what extent it is unique (at least up to sign) and it may be appropriate to consider separately the case in which J is positively or negatively oriented. In the case in which M is a symmetric space, the question has been successfully tackled by Burstall and Rawnsley [12] by introducing the twistor space of M, and in general this is a valuable tool for characterizing the existence of OCS's.

Note that an OCS J remains orthogonal if the metric g is replaced by a conformally equivalent one $e^f g$. The above problem therefore relates more accurately to the conformal class $[g]$ determined by g. Any 2-dimensional oriented conformal structure uniquely determines an integrable complex structure, so we shall always suppose that $n \geq 2$. Posing the above question in another way leads to the

Problem. Find conformal structures $(M, [g])$ which admit an abundance, at least locally, of OCS's.

We shall see that continuous families of such OCS's arise from the partial integrability of the twistor space, which is determined by properties of the Weyl conformal curvature tensor W. The study of twistor spaces of Riemannian 4-manifolds has advanced considerably since their inception in [4], and the most important aspect of the preceding problem for $n = 2$ is the classification of self-dual structures. A summary of the state of this art is included in Section 4, and provides motivation for work on higher-dimensional situations. We shall see that a pre-requisite for progress here is a more complete algebraic understanding of W, and the extent to which it may be constrained.

The next series of examples illustrate and clarify the above problems.

Example 1. (i) If $n > 1$ then the sphere S^{2n} has no OCS J relative to its standard metric. Indeed, it is well known that for $n \neq 1, 3$, the sphere S^{2n} does not even admit an *almost* complex structure. But it is also true that S^6 has no OCS (see e.g. [29]), even though it does have a G_2-invariant non-integrable acs.
(ii) Let $x \in S^{2n}$; the induced metric on $S^{2n} \setminus \{x\}$ is conformally equivalent to the flat metric on $\mathbb{R}^{2n}$. The latter admits infinitely many OCS's including the constant ones parametrized by the homogeneous space $O(2n)/U(n)$.

Example 2. (i) The complex projective space $\mathbb{CP}^n$ with the Fubini-Study metric admits a standard OCS that we denote by J_0, but there are no others apart from $-J_0$ (a result of D. Burns and the authors of [11]). Note that $\pm J_0$ induce the same orientation on $\mathbb{CP}^n$ if and only if n is even.
(ii) In fact on *any* open set $U \subset \mathbb{CP}^2$, the only OCS's inducing the standard orientation are $\pm J_0$. By contrast, if x is a point and L a projective line in $\mathbb{CP}^2$ then $\mathbb{CP}^2 \setminus \{x\}$ has exactly one pair $\pm J_x$ of negatively-oriented OCS's, and $\mathbb{CP}^2 \setminus L$ has infinitely many OCS's.

The last examples show that there is an interesting interface between the global and local existence questions. Below, we shall explain the above statements and raise a number of related questions in the course of a general survey of relevant material.

2. Curvature conditions

Let R denote the Riemann curvature tensor of g. To describe this, we fix an arbitrary point $x \in M$ and evaluate all tensors and forms at x. In this way R is an element of the vector space

$$\mathcal{R} = S^2(\Lambda^2) \ominus \Lambda^4, \tag{2}$$

where Λ^k is an abbreviation for the exterior product $\bigwedge^k T_x^*M$ and $A \ominus B$ denotes the orthogonal complement of B in A. Moreover, there is a direct sum

$$\mathcal{R} = \mathcal{S} \oplus \mathcal{W},$$

where $\mathcal{W}$ is the kernel of the Ricci contraction, and $\mathcal{S} \cong S^2 T_x^*M$ is its orthogonal complement (see e.g. [6, 44]). The component W of R in $\mathcal{W}$ is the so-called *Weyl tensor*. This is (at least up to scale) independent of the choice of metric within a conformal class, and vanishes if and only if g is locally conformally equivalent to the flat metric.

Relative to an acs J on T_xM there is a decomposition

$$(\textstyle\bigwedge^2 T_x^*M)_{\mathbb{C}} = \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2}$$

of complexified 2-forms into types. The real vector space $\mathcal{W}_J$ underlying

$$S^2(\Lambda^{2,0}) \ominus \Lambda^{4,0} \tag{3}$$

is a summand of $\mathcal{W}$ since any contraction of $\Lambda^{2,0}$ with itself is zero. It is also orthogonal to the space containing the curvature tensors of a Kähler metric, and it follows that if J extends to an orthogonal acs on a neighbourhood U of x then the component W_J of W in $\mathcal{W}_J$ is completely determined by the 'torsion tensor' ∇J and its derivative at x [19]. In fact, for algebraic reasons, only that part of ∇J represented by the Nijenhuis tensor N_J contributes to W_J, and one obtains the following interpretation of a result of Tricerri and Vanhecke [47, Theorem 11.4] (who denote $\mathcal{W}_J$ by $\mathcal{W}_7 = \mathcal{C}_5$):

Proposition 1. *If J is an OCS on an open set U then $W_J \equiv 0$ on U.*

The dimension of the space of curvature tensors is given by the well-known formula

$$\dim \mathcal{R} = \tfrac{1}{12}(2n)^2((2n)^2 - 1), \tag{4}$$

that may be deduced from (2). Thus,

$$\dim \mathcal{W} = \dim \mathcal{R} - \tfrac{1}{2}(2n)(2n+1) = \tfrac{1}{3}n(4n^3 - 7n - 3).$$

The complex dimension of (3) is found by replacing $2n$ by n in the right-hand side of (4), and so

$$\dim \mathcal{W}_J = \tfrac{1}{6} n^2(n^2-1).$$

This dimension count has an amusing consequence. Let us temporarily say that a set of k OCS's is *independent* if the respective spaces $\mathcal{W}_J$ span their maximum $k.\tfrac{1}{6}n^2(n^2-1)$ dimensions. Then because $\dim \mathcal{W} < 8 \dim \mathcal{W}_J$ for all $n \geq 2$, we have the

Corollary. *If there are 8 independent OCS's on U then (regardless of dimension) $W \equiv 0$ on U.*

It follows that there are severe restrictions on the existence of OCS's on non conformally-flat manifolds, though the corollary would be of more practical benefit if one knew equivalent characterizations of independence. The fact of the matter is that multiple OCS's tend to appear in continuous families which constrain rather than annihilate W. The situation in 4 dimensions is rather easy in this respect and is described below; that in 6 dimensions should be simpler than the general case but has not yet been studied systematically. In fact, 6 independent OCS's suffice to render M^4 conformally flat, and 7 independent OCS's render M^6 conformally flat.

For the remainder of this section we suppose that $n=2$, and thus let $M = M^4$ denote an oriented 4-dimensional Riemannian manifold. The Hodge operator $*\colon \Lambda^2 \to \Lambda^2$ gives rise to an $SO(4)$-invariant decomposition

$$\bigwedge^2 T_x^* M = \Lambda^+ \oplus \Lambda^-,$$

where $\Lambda^\pm$ is the $\pm$-eigenspace of $*$. The Weyl tensor is the sum of two components

$$W = W^+ + W^-, \qquad W^\pm \in S^2(\Lambda^\pm). \tag{5}$$

Keeping the point x fixed, we begin by analysing the tensor W^+ in more detail. Regarding W^+ as a self-adjoint transformation of the subspace Λ^+, the latter has an orthonomal basis $\{\omega_1, \omega_2, \omega_3\}$ of eigenvectors and we may suppose that

$$W^+ = a\,{\omega_1}^2 + b\,{\omega_2}^2 + c\,{\omega_3}^2,$$

where $a \geq b \geq c$. In fact W^+ has zero trace, so $c = -(a+b)$ and $a + 2b \geq 0$.

Suppose that J is an orthogonal acs on the vector space T_xM with fundamental 2-form

$$\omega = x\,\omega_1 + y\,\omega_2 + z\,\omega_3, \qquad x^2+y^2+z^2 = 1.$$

Relative to J, $\Lambda^{2,0}$ is a subspace of $(\Lambda^+)_{\mathbb{C}}$, and it follows that $\mathcal{W}_J \subset S^2(\Lambda^+)$. If $|z| = 1$ then $\Lambda^{2,0}$ is spanned by $\omega_1 \pm x\omega_2$, and $W_J = 0$ if and only if $a = b$. In the generic case $|z| < 1$ the real vector space underlying $\Lambda^{2,0}$ is spanned by

$$-y\,\omega_1 + x\,\omega_2, \qquad xz\,\omega_1 + yz\,\omega_2 + (z^2-1)\omega_3,$$

and a calculation shows that $W_J = 0$ if and only if ω or $-\omega$ is one of

$$\sqrt{\frac{a-b}{2a+b}}\omega_1 + \sqrt{\frac{a+2b}{2a+b}}\omega_3, \tag{6}$$

$$\sqrt{\frac{a-b}{2a+b}}\omega_1 - \sqrt{\frac{a+2b}{2a+b}}\omega_3. \tag{7}$$

A related result in the context of Hopf surfaces can be found in [22].

These calculations have the following consequences. Suppose that J extends to an OCS on a neighbourhood U of x. Then we may choose the sign of ω_3 so that the fundamental 2-form of J equals (6) at x. If there is an oriented OCS on U distinct from $\pm J$ at x its fundamental form must be plus or minus (7). In general though the 2-form (7) will determine an acs whose Nijenhuis tensor satisfies a certain first-order differential equation. Specializing further, the existence of three pairs of oriented OCS's on an open set U, with distinct values at x, will force W^+ to vanish at x. If W^+ vanishes identically on M, the latter is said to be *anti-self-dual* (ASD). We can therefore summarize this discussion by

Proposition 2. *Each point $x \in M^4$ is contained in an open set U on which there are zero, one, two, or infinitely many distinct pairs of positively-oriented OCS's.*

If U has exactly one pair of positively-oriented OCS's, W^+ may or may not have a repeated eigenvalue at x. If it does then the corresponding eigenvector is the fundamental 2-form of an *integrable* acs. This is the situation for a Kähler surface, and justifies the first statement in Example 2(ii). A Riemannian analogue of the Goldberg-Sachs theorem [38, 3] asserts that W^+ must also have a repeated eigenvalue on any Einstein-Hermitian surface; such metrics were first considered in [16].

Metrics which are 'doubly-Hermitian' in the sense that they admit exactly two pairs of OCS's on an open set have been described by Kobak on tori and Hopf surfaces [28]. Taking account of the opposite orientation these examples admit a total of 4 pairs of OCS's. Whether there is a direct relationship between ASD and such doubly-Hermitian metrics is as yet unclear, although the deformation theory of [27] is likely to be relevant to this study. In Section 4 we shall see that even if M is ASD it may or may not admit a *global* OCS.

3. Riemannian twistor spaces

Let J_0 denote a standard acs on the vector space $\mathbb{R}^{2n}$. The orthogonal group $O(2n)$ acts on $\mathbb{R}^{2n}$, and the stabilizer

$$\{g \in O(2n) : g \circ J \circ g^{-1} = J\}$$

of J is isomorphic to the unitary group $U(n)$. The correspondence $g \circ J \circ g^{-1} \leftrightarrow gU(n)$ is therefore a bijection between between the set of orthogonal acs's on the vector space $\mathbb{R}^{2n}$, or equivalently constant OCS's on the manifold $\mathbb{R}^{2n}$, and the coset space $O(2n)/U(n)$. The latter is isomorphic to two copies of the Hermitian-symmetric space

$$H_n = \frac{SO(2n)}{U(n)}, \tag{8}$$

corresponding to the two possible orientations of $\mathbb{R}^{2n}$.

We may regard H_n as a set totally isotropic complex n-dimensional subspaces of $\mathbb{C}^{2n}$ (endowed with an $SO(2n)$-structure). This shows that the trivial bundle $H_n \times \mathbb{C}^{2n}$ possesses a holomorphic rank n subbundle V, and we let Z^+ denote the total space of the quotient or dual bundle V^*. Consider the mapping

$$\begin{array}{ccc} \mathbb{R}^{2n} \times H_n & \stackrel{f}{\longrightarrow} & Z^+ \\ (\xi, x) & \mapsto & f_x(\xi), \end{array}$$

where f_x is the composition of the inclusion $\mathbb{R}^{2n} \subset \mathbb{C}^{2n}$ with the projection from $\mathbb{C}^{2n}$ to the fibre of V^* at x. The linearity of f_x implies that f is a bijection, and it is easy to see that f is holomorphic if we endow $\mathbb{R}^{2n} \times H_n$ with the acs J_1 defined by

$$J_1(\xi, X) = (J_x \xi, J_H X), \qquad \xi \in \mathbb{R}^{2n},\ X \in T_x H_n,$$

where J_x is the acs defined by x itself and J_H is the standard Hermitian-symmetric complex structure. Then (Z^+, J_1) is the *twistor space* of $\mathbb{R}^{2n}$; note that $\pi_2 \colon Z^+ \to H_n$ is holomorphic, and $\pi_2^{-1}(x)$ is a complex submanifold of Z^+ equivalent to $(\mathbb{R}^{2n}, J_x)$.

More generally, given a $2n$-dimensional oriented Riemannian manifold M, let $P \to M$ denote the canonical principal $SO(2n)$-bundle. One may then form the associated bundle

$$\pi \colon Z^+ = P \times_{SO(2n)} H_n \longrightarrow M,$$

each fibre $\pi^{-1}(x)$ of which parametrizes positively-oriented orthogonal acs's on the vector space $T_x M$. By construction, an orthogonal acs J on an open set U of M determines a local section $s_J \colon \pi^{-1}(U) \to U$ of Z^+. The following result was effectively proved by Atiyah, Hitchin and Singer [4] although they restricted attention to $n = 2$; here we are following the approach of [43].

Theorem 1. *The total space Z admits an acs J_1 with the property that s_J is holomorphic if and only if J is integrable.*

To make sense of the last statement, note that $s_J \colon (U, J) \to (Z, J_1)$ is a mapping between almost-complex manifolds and one says that it is holomorphic if and only if its differential is complex-linear, i.e.

$$(s_J)_* \circ J = J_1 \circ (s_J)_*.$$

One may define J_1 by choosing a suitable connection; done in the right way one sees that both Z^+ and J_1 depend only on the conformal class $[g]$ [15, 23].

An acs J on U is integrable if and only if its Nijenhuis tensor N_J vanishes. If this is the case then

$$N_{J_1}((s_J)_*X, (s_J)_*Y) = (s_J)_*N_J(X,Y) = 0$$

and, what is more, the Nijenhuis tensor N_{J_1} of J_1 vanishes along $s_J(U)$. In fact, the the value of N_{J_1} at a point $z \in Z^+$ corresponding to an acs J on T_xM can be identified with the tensor W_J of Section 2 [5, 37]. Now, as J varies along the fibre $\pi^{-1}(x) \subset Z^+$, the corresponding spaces $\mathcal{W}_J$ generate $\mathcal{W}$ (if $n \geq 3$) or the space containing W^+ (if $n = 2$). This is the essence of

Theorem 2 [4, 37]. *(Z^+, J_1) is a complex manifold if and only if (for $n \geq 3$) $W \equiv 0$, or (for $n = 2$) $W^+ \equiv 0$.*

For all n, the bundle Z^- of negatively-oriented orthogonal acs's is defined in the same way as Z^+, and if n is odd the mapping $J \mapsto -J$ induces an isomorphism $Z^+ \cong Z^-$.

Example 3. The even-dimensional sphere $S^{2n} = SO(2n+1)/SO(2n)$ with its standard conformally-flat metric has a twistor space with fibre H_n and total space

$$Z^+ = \frac{SO(2n+1)}{U(n)}.$$

This is a set of totally isotropic complex n-dimensional subspaces in $\mathbb{C}^{2n+1}$, and any such subspace may be extended uniquely to a positively-oriented isotropic $(n+1)$-subspace in $\mathbb{C}^{2n+2}$. This gives an isomorphism of complex manifolds

$$(Z^+, J_1) \stackrel{\cong}{\to} (H_{n+1}, J_H).$$

The resulting fibration $H_{n+1} \to S^{2n}$ can be described very simply; it assigns to an acs J on $\mathbb{R}^{2n+2}$ the element

$$Je \in \langle e \rangle^\perp \cong \mathbb{R}^{2n+1},$$

where e is a fixed unit vector in $\mathbb{R}^{2n+2}$. The twistor space Z^+ of S^{2n} was used in the study of harmonic maps by Calabi [13].

Referring to the last example, suppose that J is an OCS on S^{2n}. Since H_{n+1} is Kähler, the complex submanifold $s_J(S^{2n})$, and therefore S^{2n} itself, has a Kähler metric. But this is impossible unless $b_2 > 0$, i.e. $n = 1$. Thus,

Corollary. *The standard sphere S^{2n} admits no global OCS for $n \geq 2$.*

This result is only significant for $n = 3$, since it is known that there is a topological obstruction to the existence of an acs in all other cases. The above argument is equivalent to LeBrun's [29]; a completely different proof can be found in [45]. We shall return to the theme of the corollary in Section 5.

4. Anti-self-dual 4-manifolds

This section provides, for the sake of completeness, a very brief survey of results concerning ASD manifolds, i.e. oriented 4-manfolds with a conformal structure for which W^+ is identically zero. The twistor space Z^+ of M is then a complex 3-manifold, any local holomorphic section of which will define a local OCS on M. Examples with no distinguished complex structure are often referred to as *self-dual* in the literature, as they can be oriented so that $W^- \equiv 0$. On the other hand, sticking to the 'ASD' terminology emphasizes that an OCS often plays a crucial role in the construction of such a manifold, if only on an open set.

The following list contains most of the examples of ASD 4-manifolds that were known until the late 1980's:

Examples 4. (i) $\overline{\mathbb{CP}^2}$, the complex projective plane with its opposite orientation, that has signature $\sigma = -1$ and Euler characteristic $\chi = 3$. This manifold admits no global acs with the same orientation, because the potential Todd genus $(\sigma + \chi)/4$ is not an integer. But of course the manifold does admit an OCS (namely the standard one, J_0) with the opposite orientation.

(ii) $\overline{\mathbb{CP}^2} \# \overline{\mathbb{CP}^2}$, the connected sum of 2 projective planes was shown to admit an ASD metric by Poon [41]. This manifold has $\sigma = -2$ and $\chi = 4$, and again cannot admit an acs.

(iii) On a Kähler surface S the tensor W^+ is completely determined by s, so if $s \equiv 0$ then S is automatically ASD; such metrics are called scalar-flat Kähler (SFK). The obvious metric on the product of the Poincaré disk Δ with constant Gausssian curvature $K \equiv -1$ and the sphere with $K \equiv 1$ is both SFK and conformally flat. Such metrics can therefore be found on $\Sigma_g \times S^2$, where Σ_g is any Riemann surface of genus $g \geq 2$.

(iv) A simply-connected 4-manifold admits a Ricci-flat Kähler metric if and only if it is *hyper-Kähler*, meaning that Z^+ is trivialized by a family

$$\{aI + bJ + cK \;:\; IJ = K = -JI,\; a^2 + b^2 + c^2 = 1\} \cong S^2 \cong \mathbb{CP}^1, \tag{9}$$

of parallel OCS's. The torus T^4, regarded as $\mathbb{H}/\mathbb{Z}^4$ admits a flat hyper-Kähler metric, as does any K3 surface by Yau's theorem [49].

(vi) $S^1 \times S^3$, regarded as a discrete quotient of $\mathbb{H} \setminus \{0\}$, admits a conformally-flat metric that is hyper-complex, meaning that Z^+ is trivialized by an S^2-family as in (9) of integrable (but not parallel) complex structures.

Donaldson and Friedman [17] and Floer [20] then showed that under certain conditions the ASD condition is preserved by the connected sum operation. Explicit ASD structures on the connected sum $k\mathbb{CP}^2$ of k copies of the projective plane, for arbitrary $k \geq 2$, were found by LeBrun [30] by realizing this space as a 1-point compactification of a SFK metric with S^1 symmetry, which is constructed using the so-called hyperbolic ansatz and a suitable moment map. The twistor spaces of the ASD structures on $k\overline{\mathbb{CP}^2}$ are also studied in detail. Related techniques produce ASD structures on blow-ups $(S^1 \times S^3) \# k\overline{\mathbb{CP}^2}$ of Hopf surfaces [31].

These examples illustrate the fundamental

Theorem 3 [46]. *Given any oriented Riemannian 4-manifold (M,g), an ASD metric exists on the connected sum $M\,\#\,k\overline{\mathbb{CP}^2}$ for all sufficiently large k.*

More explicit versions of Theorem 3 are known when M admits at least one global OCS with the same orientation, for then the operation of connecting with $\overline{\mathbb{CP}^2}$ is realized by blowing up a point. We therefore focus now on ASD Hermitian surfaces, which are classified by their first Betti number b_1 and $2\chi+3\sigma=c_1{}^2$:

Proposition 3 [7, 39]. *Let M be a compact ASD Hermitian surface.*
(i) If b_1 is even then M has a SFK metric;
(ii) If $c_1{}^2=0$ then M is a Hopf surface ($b_1=1$), or M is a torus ($b_1=4$) or a K3 surface ($b_1=0$);
(iii) If $c_1{}^2<0$ then M is a surface of class VII ($b_1=1$), or a ruled surface ($b_1=2g$).

It follows that any compact SFK surface with $c_1\neq 0$ (over $\mathbb{R}$) must be ruled. Moreover,

Theorem 4 [25]. *A SFK metric exists on some blow-up of any ruled surface.*

Building on [32], Kim and Pontecorvo prove the following [26]. If M is a compact SFK surface with $c_1\neq 0$ then *any* blow-up of M is SFK, unless $M=\mathbb{P}(L\oplus\mathcal{O})$ is a split projective bundle over a Riemann surface Σ_g with $s\geq 2$ and $\deg L=0$. A corollary is that there exists a SFK metric on the smooth 4-dimensional manifold $(\Sigma_g\times\mathbb{CP}^1)\#\overline{\mathbb{CP}^2}$ for $g>1$, even though it is known that the blow-up of the product surface $\Sigma_g\times\mathbb{CP}^1$ at 1 point cannot be SFK.

The determination of the optimal value of k in Theorem 3 outside the SFK regime is a hard problem, though certain estimates are possible using gluing techniques. For example, LeBrun and Singer prove that $\mathbb{CP}^2\#\,k\overline{\mathbb{CP}^2}$ has an ASD metric for all $k\geq 14$ [33]. Another important area in which progress has been made is the classification of ASD metrics admitting a symmetry group of dimension 2 or more [42, 24].

Not only does the above theory provide guidance as to what to study in higher dimensions, but the twistor space Z^+ of a Riemannian 4-manifold M is itself a 6-dimensional almost-complex manifold worthy of study in its own right. For example, it is a consequence of Taubes' theorem that any finitely presentable group is the fundamental group of a compact complex 3-manifold Z^+. Moreover, (Z^+,J_1) always admits a 1-parameter family of almost-Hermitian metrics g_t each of which renders the fibration

$$\pi\colon Z^+\longrightarrow M \tag{10}$$

a Riemannian submersion [35, 40]. In certain situations, Z^+ admits a number of natural OCS's which are sections of its own twistor space with fibre $H_3\cong\mathbb{CP}^3$.

Example 5. Suppose that M is hyper-complex, with an S^2-family (9) of OCS's trivializing Z^+. The latter can then be identified with the product $S^2 \times M$, and itself admits an S^2-family of OCS's

$$(aI + bJ + cK) + J_F, \tag{11}$$

formed by taking the product of a structure on M with the standard complex structure J_F on the fibre $S^2 \cong \mathbb{CP}^1$. The complex structures (11) trivialize a subbundle $P \cong Z^+ \times \mathbb{CP}^1$ of the twistor space of Z^+, and are all inequivalent to the acs J_1 [48]. The latter is defined as in (11) but now regarding a, b, c as fibre coordinates, and s_{J_1} is a non-constant section of P. It is known that (Z^+, J_1) cannot admit a Kähler metric, even if M does, though the metrics g_t are balanced [22].

5. Higher-dimensional examples

Let (M, g) be a Riemannian manifold of dimension $2n \geq 6$. If we rephrase the local version of the first problem in the Introduction in terms of twistor spaces, it makes sense to ask whether, given (M^{2n}, g), there exist complex subbundles of Z^+ that are complex submanifolds relative to J_1. Recall that if (M, g, J) is Hermitian then $s_J(M)$ lies in the zero set

$$Z_0^+ = \{z \in Z^+ : N_{J_1}(z) = 0\} \tag{12}$$

of the Nijenhuis tensor of J_1. This leads to the

Problem. Determine Z_0^+ for a given Riemannian manifold (M, g), and investigate any general properties of this subset.

On an arbitrary almost-complex manifold there is little that can be said about the zero set of the Nijenhuis tensor; in particular it need not be complex-analytic. However, there is some evidence that the subset Z_0^+ of a general twistor space is better behaved, and an important result in this direction is provided by

Theorem 5 [12]. *If G/H is a symmetric space of positive or negative type with* $\operatorname{rank} G = \operatorname{rank} H$*, then Z_0^+ is a complex submanifold of Z, each connected component of which is a flag manifold or domain.*

If we identify a tangent space $T_x(G/H)$ with an $\operatorname{Ad} H$-invariant complement $\mathfrak{p}$ of $\mathfrak{h}$ in $\mathfrak{g}$, then any point z of $\pi^{-1}(x) \subset Z^+$ corresponds to a maximal isotropic subspace $\mathfrak{p}_+$ of $\mathfrak{p}_{\mathbb{C}}$. The key point in the proof of Theorem 5 is that $z \in Z_0^+$ if and only if

$$[[\mathfrak{p}_+, \mathfrak{p}_+], \mathfrak{p}_+] \subseteq \mathfrak{p}_+, \tag{13}$$

and this converts the problem into a purely algebraic one.

The following illustration of Theorem 5 had its origin in the study of harmonic maps of surfaces to complex projective space $\mathbb{CP}^n$ [18], and belongs to a class of twistor spaces considered in [9, 43].

Example 6. The projective holomorphic tangent bundle $F=\mathbb{P}(T^{1,0}\mathbb{CP}^n)$ may be regarded as a subbundle of $Z^\pm$ over $\mathbb{CP}^n$ with fibre $\mathbb{CP}^{n-1}\subset H_n$ as follows. A point of F is a complex line $L\subset T^{1,0}_x\mathbb{CP}^n$, and this determines the acs

$$J_L=\begin{cases} J_0 \text{ on } L,\\ -J_0 \text{ on } L^\perp\end{cases} \tag{14}$$

at x. As a complex manifold, (F,J_1) is the flag manifold parametrizing pairs (V_1,V_2) where V_k is a k-dimensional subset of $\mathbb{C}^{n+1}$ and $V_1\subset V_2$. However, (14) implies that the twistor fibration $\pi\colon F\to\mathbb{CP}^n$ is then given by

$$\pi(V_1,V_2)=V_2\ominus V_1. \tag{15}$$

Extending the argument following Example 3 yields

Theorem 6 [11]. *If J is an OCS on a compact symmetric space G/H with $\operatorname{rank}G=\operatorname{rank}H$ then G/H is Hermitian and J is G-invariant.*

The hypotheses of Theorems 5 and 6 do not require G/H to be irreducible, since the theory is compatible with de Rham decompositions. This is no longer true however when $\operatorname{rank}G>\operatorname{rank}H$ [10]; a counterexample to Theorem 6 is furnished by the Calabi-Eckmann OCS's on products of odd-dimensional spheres, but its validity in some irreducible cases remains open.

The statement in Example 2(i) is a corollary of Theorem 6. Now let $(z_1,z_2)\in\mathbb{C}^2$, and define

$$\begin{aligned} V_1 &= \langle(1,z_1,z_2)\rangle\,,\\ V_2 &= \langle(1,z_1,z_2),(1,0,0)\rangle\,. \end{aligned}$$

Then $(V_1,V_2)\in F$ provided that $(z_1,z_2)\neq(0,0)$, and

$$\pi(V_1,V_2)=\big\langle(|z_1|^2+|z_2|^2,-z_1,-z_2)\big\rangle\in\mathbb{CP}^2.$$

This defines a J_1-holomorphic section of F over $\mathbb{CP}^2\setminus(\{x\}\sqcup L)$ where $x=\langle(1,0,0)\rangle$ and

$$L=\{\langle(0,z_1,z_2)\rangle : z_1,z_2\in\mathbb{C}\}.$$

Moreover, s extends holomorphically over L. The OCS J_x on $\mathbb{CP}^2\setminus\{x\}$ for which $s=s_{J_x}$ is formed by reversing the standard one J_0 according to the formula (14) on radial directions emanating from x. It follows that J_x is invariant by the isometry group $U(2)$ stabilizing x.

An elaboration of this method establishes the remaining statements in Example 2(ii). In particular, $\pm J_x$ are the only negatively-oriented OCS's on $\mathbb{CP}^2\setminus\{x\}$. Corresponding to this is the fact that s_{J_x} extends to the unique divisor D of a standard line bundle over F such that $\pi^{-1}(x)\subset D$ (the author is grateful to M. Pontecorvo for this observation). Analogous results will hold for $\mathbb{CP}^n$, and combined with Theorem 6, these remarks lead to the

Problem. Characterize H-invariant OCS's on appropriate open sets of Hermitian symmetric spaces G/H.

Suppose that M is a 4-manifold with an OCS with fundamental 2-form ω. Using projections given in [47], one has

$$W^+(\omega^2) = \tfrac{1}{12}(3s^* - s),$$

where s^* is the so-called $*$ scalar curvature. The formulae (6),(7) then relate the left-hand side directly to the eigenvalues of W^+. A more significant result of this genre was found by Gauduchon by integrating a Weitzenbock formula:

Theorem 7 [21]. *Let M be a compact Riemannian $2n$-manifold with scalar curvature s, and let c denote the smallest eigenvalue of W on $\bigwedge^2 T^*M$. If M admits an OCS J then*

$$\int_M [(n-1)s - n(2n-1)c] * 1 \;\geq\; 0.$$

Given the OCS J, equality in the last formula implies that its fundamental form ω is an eigenvector of W corresponding to c. In addition, the resulting Hermitian structure is *balanced*, which means that $d(\omega^{n-1}) = 0$ and is the next best thing to being Kähler (see [2, 34]). Theorem 7 implies that if J is an OCS on a compact quotient of a symmetric space G/H of negative type then J necessarily arises from a G-invariant complex structure on G/H which is therefore Hermitian [21].

Given the very extensive analysis that has been carried out in the 4-dimensional case, it is natural to divert some attention to the theory of OCS's on 6-manifolds. The following illustrations concern more non-Kähler manifolds.

Example 7. It is well known that the sphere S^6 admits a G_2-invariant non-integrable acs J, which relates to the above discussion of twistor spaces as follows. The fibre H_3 of Z^+ over S^6 is isomorphic to $\mathbb{CP}^3$. If one reduces the structure group $SO(7)$ to G_2, the subbundle of Z^+ complementary to s_J with fibre $\mathbb{CP}^2$ is isomorphic to the complex quadric

$$G_2/U(2) \cong \frac{SO(7)}{SO(2)\times SO(5)} \subset \mathbb{CP}^6.$$

This 'reduced' twistor space was exploited by Bryant [8] in the description of J-holomorphic curves on S^6.

Example 8. A compact even-dimensional nilmanifold can only admit a Kähler metric if it is a torus (see e.g. [14]). One of the simplest 6-dimensional examples is the Iwasawa manifold N, defined as a finite quotient of the complex Heisenberg group. There is a fibration

$$\pi\colon N \longrightarrow T^4$$

of N onto a 4-torus with fibre a 2-torus T^2, which to some extent mimics (10), although π^* is no longer injective on $H^2(T^4,\mathbb{R})$. In any case N has, in addition to its standard complex structure J_0, a 2-sphere S^2 of positively-oriented complex structures defined as in (11) [1]. The resulting Hermitian structures are all balanced, but do not correspond to eigenvectors of the Weyl tensor W (cf. Theorem 7).

The last example leads us to conclude with the following remarks. Let M be an oriented Riemannian 6-manifold. As a first step to describing the zero set (12), it is natural to investigate its possible intersection

$$Z_0^+ \cap \pi^{-1}(x) \subseteq \mathbb{CP}^3$$

with a given fibre of Z^+. In the examples so far presented, the form of this intersection is independent of x and is one of

$$\mathbb{CP}^3, \quad \mathbb{CP}^2 \sqcup \{z\}, \quad \mathbb{CP}^1 \sqcup \mathbb{CP}^1, \quad \mathbb{CP}^1 \sqcup \{z\},$$

where $\mathbb{CP}^k$ is a *linear* subspace of $\mathbb{CP}^3$ and z is a point corresponding to some standard isolated OCS.

By contrast, using (13), one can show that the complex 3-dimensional quadric

$$Q \cong \frac{SO(5)}{SO(2)\times SO(3)} \cong \frac{Sp(2)}{U(2)}$$

in $\mathbb{CP}^4$ has

$$Z_0^+ = s_{J_H}(Q) \sqcup F, \qquad F \cong \frac{SO(5)}{T^2},$$

where J_H is its Hermitian-symmetric complex structure, and

$$Z_0^+ \cap \pi^{-1}(x) = \{z\} \sqcup C,$$

where $C \cong \mathbb{CP}^1$ is a *conic* in the complement $\mathbb{CP}^2$ of z. Finally, we saw in Section 2 that a Riemannian 6-manifold M can conceivably admit 6 independent isolated OCS's, and it would be interesting to know whether or not such examples can be realized.

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References

1. E. Abbena, S. Garbiero, S. Salamon: Hermitian geometry on the Iwasawa manifold, Boll. Un. Mat. Ital., to appear.
2. L. Alessandrini, G. Bassanelli: Metric properties of manifolds bimeromorphic to compact Kähler spaces, J. Differ. Geom. **37** (1993), 95–121.
3. V. Apostolov, P. Gauduchon: The Riemannian Goldberg-Sachs theorem, preprint.
4. M.F. Atiyah, N.J. Hitchin, I.M. Singer: Self-duality in four-dimensional Riemannian geometry, Proc. Roy. Soc. Lond. **A362** (1978), 425–461.

5. L. Bérard Bergery, T. Ochiai: On some generalizations of the construction of twistor space, in Global Riemannian Geometry, eds. T. Willmore, N. Hitchin, Ellis Horwood, 1984.
6. A. Besse: Einstein manifolds, Springer, 1987.
7. C.P. Boyer: Conformal duality and compact complex surfaces, Math. Ann. **274** (1986), 517–526.
8. R.L. Bryant: Submanifolds and special structures on the octonians, J. Differ. Geom. **17** (1982), 185–232.
9. R.L. Bryant: Lie groups and twistor spaces, Duke Math J. **52** (1985), 223–261.
10. F. Burstall, S. Gutt, J. Rawnsley: Twistor spaces for Riemannian symmetric spaces, Math. Ann. **295** (1993), 729–743.
11. F. Burstall, O. Muškarov, G. Grantcharov, J. Rawnsley: Hermitian structures on Hermitian symmetric spaces, J. Geom. Phys. **10** (1993), 245–249.
12. F.E. Burstall, J.H. Rawnsley: Twistor theory for Riemannian symmetric spaces, Lect. Notes Math. 1424, Springer, 1990.
13. E. Calabi: Minimal immersions of surfaces in Euclidean spheres, J. Differ. Geom. **1** (1967), 111–125.
14. L.A. Cordero, M. Fernández, A. Gray: Symplectic manifolds without Kähler structure, Topology **25** (1986), 375–380.
15. P. de Bartolomeis, L. Migliorini, A. Nannicini: Propriétés globales de l'espace de twisteurs, Rend. Mat. Acc. Lincei **9** (1991), 147–153.
16. A. Derdziński: Self-dual Kähler manifolds and Einstein manifolds of dimension four, Compositio Math. **49** (1983) 405–433.
17. S.K. Donaldson, R. Friedman: Connected sums of self-dual manifolds and deformations of singular spaces, Nonlinearity **2** (1989), 197–239.
18. J. Eells, J.C. Wood: Harmonic maps from surfaces to complex projective spaces, Adv. Maths. **49** (1983), 217–263.
19. A. Farinola, M. Falcitelli, S. Salamon: Almost-Hermitian geometry, Diff. Geom. Appl. **4** (1994) 259–282.
20. A. Floer: Self-dual conformal structures on ℓCP^2, J. Differ. Geom. **33** (1991), 551–573.
21. P. Gauduchon: Structures complexes sur une varieté conforme de type negative, in Complex Analysis and Geometry, Lect. Notes Pure Appl. Math. 173, Marcel Dekker, 1995.
22. P. Gauduchon: Structures de Weyl-Einstein, espaces de twisteurs et variétés de type $S^1 \times S^3$, to appear in J. Reine Angew. Math.
23. Y. Inoue: Twistor spaces of even-dimensional Riemannian manifolds, J. Math. Kyoto Univ. **32** (1992), 101–134.
24. D.D. Joyce: Explicit construction of self-dual 4-manifolds, Duke Math. J. **77** (1995), 519–552.
25. J. Kim, C. LeBrun, M. Pontecorvo: Scalar-flat Kähler surfaces of all genera, preprint.
26. J. Kim, M. Pontecorvo: A new method of constructing scalar-flat Kähler surfaces, J. Differ. Geom. **41** (1995), 449–477.
27. A.D. King, D. Kotschick: The deformation theory of anti-self-dual conformal structures, Math. Ann. **294** (1992), 591–609.
28. P.Z. Kobak: Explicit doubly-hermitian metrics, ESI preprint 197, Vienna, 1995.
29. C. LeBrun: Orthogonal complex structures on S^6, Proc. Am. Math. Soc. **101** (1987), 136–138.
30. C. LeBrun: Explicit self-dual metrics on $\mathbb{CP}_2\#\cdots\#\mathbb{CP}_2$, J. Differ. Geom. **34** (1991), 223–253.
31. C. LeBrun, Anti-self-dual Hermitian metrics on the blown-up Hopf surface, Math. Ann. **289** (1991) 383–392.
32. C. LeBrun, M. Singer: Existence and deformation theory for scalar-flat Kähler metrics on compact complex surfaces, Invent. Math. **112** (1993), 273–313.
33. C. LeBrun, M. Singer: A Kummer-type construction of self-dual 4-manifolds, Math. Ann. **300** (1994), 165–180.
34. M.L. Michelsohn: On the existence of special metrics in complex geometry, Acta Math. **149** (1982), 261–295.
35. O. Muškarov, Structures presques hermitiennes sur des espaces twistoriels et leur types, C. R. Acad. Sci. Paris **305** (1987), 307–309.
36. A. Newlander, L. Nirenberg: Complex analytic coordinates in almost complex manifolds, Ann. of Math. **65** (1957), 391–404.

37. N.R. O'Brian, J.H. Rawnsley: Twistor spaces, Ann. Glob. Anal. Geom. **3** (1985), 29–58.
38. R. Penrose, W. Rindler: Spinors and Space-time, Vol. 2, Cambridge University Press, 1988.
39. M. Pontecorvo: On twistor spaces of anti-self-dual Hermitian surfaces, Trans. Am. Math. Soc. **331** (1992), 653–631.
40. M. Pontecorvo: On the complex geometry of twistor spaces, preprint.
41. Y.S. Poon: Torus symmetry of compact self-dual manifolds, preprint.
42. Y.S. Poon: Conformal transformations of compact self-dual manifolds, Intern. J. Math **5** (1994), 125–140.
43. S. Salamon: Harmonic and holomorphic maps, in Seminar Luigi Bianchi II, Lect. Notes Math. 1164, Springer, 1985.
44. S. Salamon: Riemannian Geometry and Holonomy Groups, Pitman Research Notes Math. 201, Longman, 1989.
45. K. Sekigawa, L. Vanhecke: Almost Hermitian manifolds with vanishing first Chern classes, Rend. Sem. Mat. Univer. Politec. Torino **50** (1992), 195–208.
46. C.H. Taubes: The existence of anti-self-dual conformal structures, J. Differ. Geom. **36** (1992), 163–254.
47. F. Tricerri, L. Vanhecke: Curvature tensors on almost Hermitian manifolds, Trans. Am. Math. Soc. **267** (1981), 365–398.
48. V.V. Tsanov: Moduli of twistor spaces, Proc. Conf. Differential geometric methods in theoretical physics, World Scientific, 1986.
49. S.T. Yau: On the Ricci-curvature of a complex Kähler manifold and the complex Monge-Ampère equations, Comment. Pure Appl. Math. **31** (1978) 339–411.