COMPLEX STRUCTURES AND TWISTORS

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Based on recent work of

[BV] Borisov-Viaclovsky
[BSV] Borisov-Salamon-Viaclovsky
[PS] Povero-Salamon
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and earlier work of

[B] Bishop 1964
[L] LeBrun 1986
[P] Pontecorvo 1992
[S] Slupinsky 1996
[W] Wood 1992

A Kähler manifold

$$
\begin{aligned}
Z_n &= \frac{SO(2n)}{U(n)} \\
&= \big\{\text{orthogonal acs's on } (\mathbb{R}^{2n}, +)\big\} \\
&= \big\{J \in \mathfrak{o}(2n) \cap O(2n) : \operatorname{Pf} J = 1\big\}\,;
\end{aligned}
$$

recall $\det X = (\operatorname{Pf} X)^2$ for X skew-symmetric.

Any two of

$$J + \widehat{J} = 0, \quad J\widehat{J} = I, \quad J^2 = -I.$$

implies the third.

$$n = 1 \quad \Rightarrow \quad J = \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Complex coordinates

$$Z_n = \{\,\text{maximal isotropic subspaces } \Lambda \subset \mathbb{C}^{2n}\,\},$$

where Λ should be thought of as $\Lambda^{1,0} \subset (T^*_x M) \otimes_{\mathbb{R}} \mathbb{C}$.

On a dense open set of Z_n,

$$\Lambda = \Big\langle dz^i + \sum_j \zeta^i_j d\overline{z}^j \ : \ i = 1, \dots n \Big\rangle$$

with $\zeta^i_j + \zeta^j_i = 0$ for orthogonality.

$$\Longrightarrow \ \dim_{\mathbb{C}} Z_n = \binom{n}{2} = 0, 1, 3, 6, 10, \dots$$

A complete atlas to cover Z_n requires spinors.

Pure spinors

$\mathrm{Spin}(2n)$ acts on $\Delta = \Delta_+ \oplus \Delta_-$, with

$$\Delta_+ \otimes \Delta_+ \;\cong\; \Lambda^n_+ \oplus \Lambda^{n-2} \oplus \Lambda^{n-4} \oplus \cdots$$

$\delta \in \Delta_+$ is 'pure'

$\iff$ $\ker(\delta\colon \mathbb{C}^{2n} \to \Delta_-)$ is maximal $(= \Lambda)$

$\iff$ $\delta \otimes \delta \in \Lambda^n_+$.

$$Z_1 = \mathrm{pt}, \quad Z_2 = \mathbb{P}^1, \quad Z_3 = \mathbb{P}^3, \quad Z_4 = Q^6 \subset \mathbb{P}^7.$$

For $n \geqslant 5$, we get an intersection of quadrics:

$$Z_n = \bigcap_i Q_i \subset \mathbb{P}(\Delta_+) = \mathbb{P}^N, \quad N = 2^{n-1} - 1.$$

Twistor fibration

$$Z_n = Z^{\binom{n}{2}} = Z(\mathbb{R}^{2n})$$

A decomposition $\mathbb{R}^{2n+2} = \langle e_0 \rangle \oplus \mathbb{R}^{2n+1}$, equivalently a reduction from $SO(2n+2)$ to $SO(2n+1)$, gives

$$\begin{array}{ccc} & Z_{n+1} & \ni J \\ \pi & \downarrow & \\ & S^{2n} & \ni J(e_0) \end{array}$$

Each fibre $\pi^{-1}(x) = Z(T_x S^{2n}) = Z_n$ parametrizes acs's on $T_x S^{2n}$ and is a complex submanifold of Z_{n+1}.

A local section $s: U \rightarrow Z_{n+1}$ defines an orthogonal almost complex structure on $U \subset S^{2n}$.

Over the six-sphere

$$\begin{array}{ccc} & Z_4 = Q^6 & \\ \mathbb{P}^3 & \downarrow & \\ & S^6 & \supset\ \mathbb{R}^6 \end{array}$$

An orthogonal complex structure on S^6 would have a graph Γ that is holomorphic, inducing a Kähler metric on S^6, impossible [L].

But does $\mathbb{R}^6$ admit a non-constant complex structure, compatible with the conformally flat metric?

Bear in mind that any such OCS on $\mathbb{R}^4$ is constant and so equals J_λ with $\lambda \in Z_1 = \mathbb{P}^1$ [W].

The twistor space of $\mathbb{R}^4$

YES! $\mathbb{R}^6$ does admit a non-constant OCS. View $\mathbb{R}^6$ as a subset of $\mathbb{P}^3$ via the conformal map $\mathbb{R}^2 \subset \mathbb{P}^1$:

$$\begin{array}{ll} \mathbb{P}^3 & \supset \mathbb{R}^4 \times \mathbb{P}^1 \supset \mathbb{R}^4 \times \mathbb{R}^2 = \mathbb{R}^6 \\ \downarrow & \\ S^4 & \supset \mathbb{R}^4 \end{array}$$

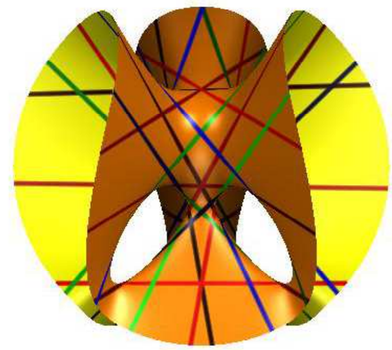
Given an algebraic surface $S \subset \mathbb{P}^3$ of degree d, let f be the number of fibres (j -invariant skew lines) it contains over S^4 or $\mathbb{R}^4$. Then

$d = 1 \implies f = 1.$

$d = 2 \implies f = 0, 1, 2$ or ∞.

S irreducible $\implies f \leqslant d^2$.

Cubics and quadrics in $\mathbb{P}^3$



A non-singular cubic has at most 6 skew lines.

Theorem [PS] If $d=3$ and $f \geqslant 5$ then S is reducible.

If S has 5 fibres $L_i = \pi^{-1}(x_i)$ they must all meet 2 other lines $L_6,\ L_7 = j(L)$, but then $\dim \langle L_1, \ldots, L_5 \rangle = 4$ and x_i lie on a circle.

$$\implies S = Q \cup \mathbb{P}^2, \qquad Q \cong S^2 \times S^2.$$

Moreover,

$$Q \setminus (S^1 \times S^2) \xrightarrow{2\times 1} S^4 \setminus S^1 = \Omega$$

defines OCS's $\pm J$ on Ω.

Theorem [SV] If J is an OCS on S^4 minus a round circle then J arises from a j-invariant quadric in $\mathbb{P}^3$.

The generic quadric in $\mathbb{P}^3$

By contrast,

Theorem [SV] A quadric with $f = 0$ will touch the fibres over a smooth torus $S^1 \times S^1 = T$ and

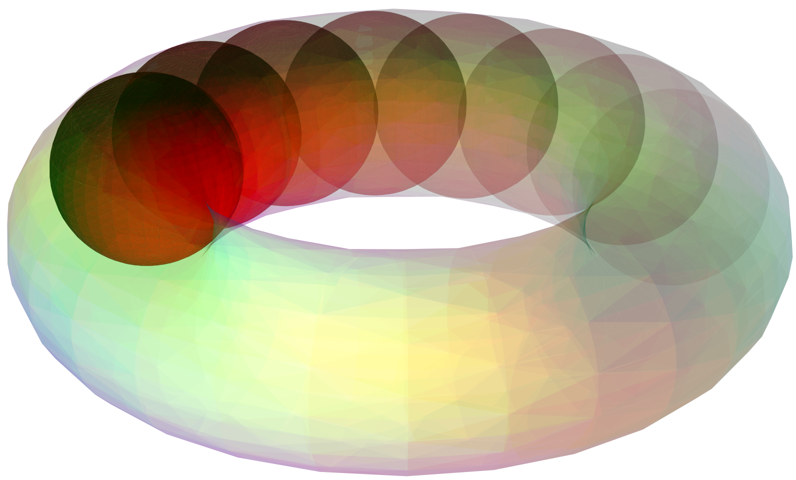
$$Q \setminus (S^1 \times S^2) \xrightarrow{2\times 1} S^4 \setminus \mathbb{T}$$

is a bihermitian structure (J, J') on S^4 minus a solid torus $\mathbb{T}$.

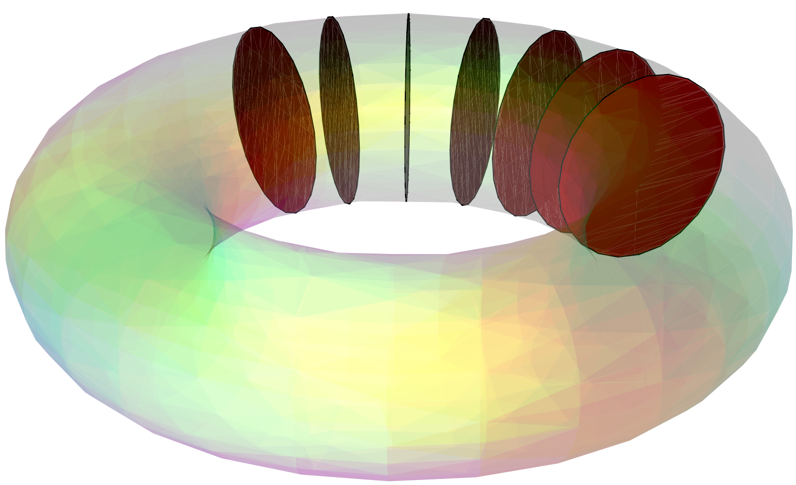
Problem: characterize the conformal embeddings of T in S^4 that arise in this way.

Cf. the projection of any holomorphic curve in $\mathbb{P}^3$ is a Willmore surface in S^4.

A solid torus glued to itself



$S^1 \times S^2 \subset \mathbb{P}^3$



$\mathbb{T} \subset S^4$

Twisted or warped product OCS's

Recall that $Z_2 = Z(\mathbb{R}^4) = \mathbb{P}^1$.
Any meromorphic $w\colon \mathbb{C} \to \mathbb{C}$ defines an OCS

$$\begin{aligned} (z_1, z_2, z_3) \in \mathbb{R}^6 &= \mathbb{R}^4 \times \mathbb{R}^2 \\ \mathbb{J} &= (J_{w(z_3)},\ J_0). \end{aligned}$$

Taking $w(z_3) = z_3$ gives the twistor space of $\mathbb{R}^4$.

If w is rational, we shall explain that the graph of $\mathbb{J}$ extends to an algebraic 3-fold in $Q^6 \subset \mathbb{P}^7$.

By contrast, a doubly-periodic function like $w = \wp$ induces a non-standard OCS on T^6.

Problem: Classify conformally-flat Hermitian manifolds of real dimension 6, cf. [P].

Bishop's theorem

$$\begin{array}{ccc} \mathbb{P}^3_\infty \subset & Q^6 & \\ & \downarrow & \\ \infty \in & S^6 & \supset \ \mathbb{R}^6 = \mathbb{R}^4 \times \mathbb{R}^2 \end{array}$$

Any constant OCS on $\mathbb{R}^6$ ($w = c$) arises from a $\mathbb{P}^3_{\rm hor}$ that intersects $\mathbb{P}^3_\infty$ in a $\mathbb{P}^2$. These OCS's are correctly parametrized by $(\mathbb{P}^3)^* \cong Z_3$.

If w is rational the graph Γ of $\mathbb{J}$ has finite 'energy', characterized by Hausdorff 6-measure or a tensor norm over S^6:

$$Hf^6(\Gamma) < \infty \quad \Longleftrightarrow \quad \|\nabla J\|^6_{S^6} < \infty.$$

$\overline{\Gamma}$ is an analytic subset of $\mathbb{P}^7$ [B], so algebraic.

Also in this case, $\overline{\Gamma} \cap \mathbb{P}^3_\infty = \mathbb{P}^2 \ \ldots$

Main result in dimension 6

Theorem [BSV] Any finite-energy OCS on $\mathbb{R}^6$ (or on $\mathbb{R}^6$ minus n points) is isometric to a warped OCS $\mathbb{J}$ associated to some rational function $w\colon z_3 \mapsto J_{z_3}$.

The proof relies on a classification of 3-folds X in Q^6 of bidegree $(p, 1)$, meaning

$$X \cdot \mathbb{P}^3_{\text{hor}} = p, \qquad X \cdot \mathbb{P}^3_{\text{ver}} = 1.$$

The analogous theorem is open on $\mathbb{R}^{2n}$ with $n \geqslant 4.$

A $\mathbb{P}^3_{\text{ver}}$ is either a fibre over $S^6,$ or the twistor space of a conformal S^4 in S^6 [S].

The graph Γ of $\mathbb{J}$

Given $q=[\mathbf{x},\mathbf{y}]\in\mathbb{P}(\mathbb{C}^4\oplus\mathbb{C}^4)$, define Q^6 by $\mathbf{x}\,\widehat{\mathbf{y}}=0$.

Assume $\mathbf{x}\neq 0\neq\mathbf{y}$. Then $q\in Q^6$ iff $\mathbf{x}=\mathbf{y}M$ for some unique

$$M=\begin{pmatrix} 0 & -z_3 & -z_2 & -z_1\\ z_3 & 0 & -\overline{z}_1 & \overline{z}_2\\ z_2 & \overline{z}_1 & 0 & -\overline{z}_3\\ z_1 & -\overline{z}_2 & \overline{z}_3 & 0\end{pmatrix}\in\mathbb{R}^+\times(\mathfrak{o}(4,\mathbb{C})\cap SU(4))$$

defining $\pi(q)\in\mathbb{C}^3\setminus\{0\}\subset S^6$.

If $\mathbb{J}$ is a warped OCS defined by w,

$$\Gamma=\Big\{[\mathbf{x},M\mathbf{y}]:\mathbf{x}=(1,0,0,w(z_3))\Big\}$$

lies in a singular 4-quadric $Q^4_s\subset\mathbb{P}^5$ of rank 4. Here, $\mathbf{x}$ represents $J_{z_3}\in\mathbb{P}^1\subset\mathbb{P}^3$.

Classification of 3-folds

Theorem [BV] An irreducible 3-fold X of bidegree $(1,p)$ in Q^6 is one of:

- a horizontal $\mathbb{P}^3$ ($p=0$),
- a smooth 3-quadric Q^3 ($p=1$),
- the cone over a Veronese $\mathbb{P}^2 \subset Q^4$ ($p=3$),
- a Weil divisor in a rank 4 quadric Q^4_s ($p \geqslant 1$).

A Q^3 example arises from the action of G_2 on S^6, and defines an OCS on $S^6 \setminus S^2 \cong S^3 \times H^3$.

We require $\pi : X \to S^6$ to be $1:1$ except over ∞, and $\pi^{-1}(\infty) = \mathbb{P}^3_\infty$ must intersect X in a $\mathbb{P}^2$ that contains a line $\mathbb{P}^1 = L$ singular in X. This rules out cases 2 and 3. There is in fact a projection

$$X \setminus L \longrightarrow \mathscr{C} \subset \mathbb{P}^1 \times \mathbb{P}^1.$$

We can then reconstruct the warped example.

The situation in higher dimensions

$$\begin{array}{ccc} & Z_{n+1} & \supset Z_{n+1} \setminus Z_n \\ \pi & \downarrow & \\ & S^{2n} & \supset \ \mathbb{R}^{2n} \end{array}$$

Given an OCS of finite energy on $\mathbb{R}^{2n}$, all the interest is concentrated on $\overline{\Gamma} \setminus \Gamma \subset Z_n$, and may define an OCS on $\mathbb{R}^{2n-2}$.

But beware: $\dim H_{2n}(Z_{n+1}, \mathbb{R}) \sim p_n$ is large!

Theorem [BSV] An OCS J on $\mathbb{R}^{2n}$ asymptotically constant: (meaning $\|J(\mathbf{x}) - J_0\| \to 0$ as $\mathbf{x} \to \infty$) then $\overline{\Gamma} \setminus \Gamma = \mathbb{P}^{n-1}$, and $J = J_0$ is actually constant.