Orthogonal Complex Structures and Twistor Surfaces

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Definitions and examples

Let Ω be an open set of $\mathbb{R}^4$. An OCS on an open set $\Omega \subset \mathbb{R}^4$ is a (C^1) map

$$J : \Omega \to \{M \in SO(4) : M^2 = -I\} \cong \mathbb{CP}^1$$

satisfying $Nij(J) \equiv 0$.

J_0 is defined on $\mathbb{R}^4 \cong \mathbb{C}^2$ by fixing complex coordinates z_1, z_2.

J_1 is defined on $\mathbb{R}^4 \setminus \{0\}$ by applying to J_0 the map $q \mapsto 1/q$ where $q = z_1 + jz_2 \in \mathbb{H}$.

J_2 is defined on

$$\mathbb{R}^4 \setminus \ell \;\cong\; S^2 \times \mathbb{R}^+ \times \mathbb{R} \;\cong\; S^2 \times H^2,$$

a domain with a conformally-flat complete Kähler metric (Pontecorvo 1992).

Liouville theorems

Let J be an OCS on an open set $\Omega = \mathbb{R}^4 \setminus K$.

$\boxed{\boxed{1}}$ If $\mathcal{H}^1(K) = 0$ (Hausdorff measure) then J is defined on $\mathbb{R}^4$ or $\mathbb{R}^4 \setminus \{p\}$ and is at least conformal to J_0.

$\boxed{\boxed{2}}$ If K is a straight line or (equivalently) a round circle then J is conformal to J_0 or J_2.

Theorem 1 was known, at least when $K = \emptyset$. If J is defined on $\mathbb{R}^4$ then it must be constant.

Proofs will use the Penrose twistor space $\mathbb{CP}^3$ fibring to $S^4 = \mathbb{R}^4 \cup \{\infty\}$.

The language of deformations

Start with J_0 and $\Lambda^{1,0}(J_0) = \begin{cases} dz_1 \\ dz_2. \end{cases}$

Define J by the isotropic family

$$\Lambda^{1,0}(J) = \begin{cases} dz_1 - a\,d\overline{z}_2 \\ dz_2 + a\,d\overline{z}_1 \end{cases} = \begin{cases} dW_1 \\ dW_2 \end{cases} \text{ mod } da,$$

where $W_1 = z_1 - a\overline{z}_2$ and $W_2 = z_2 + a\overline{z}_1$ and

$$a = a(z_1, z_2) = a(W_1, W_2).$$

Equivalently, $J = P_\phi J_0 P_\phi^{-1}$, where

$$\phi = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \in \Gamma(\Theta \otimes \Lambda^{0,1}).$$

The integrability condition is that a be a holomorphic function of W_1, W_2; explicitly

$$\frac{\partial a}{\partial \overline{z}_1} - a\frac{\partial a}{\partial z_2} = 0 = \frac{\partial a}{\partial \overline{z}_2} + a\frac{\partial a}{\partial z_1}.$$

The language of quaternions

Let $W_1 = z_1 - a\bar{z}_2$ and $W_2 = z_2 + a\bar{z}_1$. A holomorphic function $a : \Omega \to \mathbb{C}\cup\{\infty\}$ defines a surface

$$\begin{array}{ccc}
\Big\{[1, a, W_1, W_2] : z_1, z_2 \in \mathbb{C}\Big\} & \subset & \mathbb{CP}^3 \setminus \mathbb{CP}^1 \\
 & & \Big\downarrow \pi \\
\Big\{[1+ja,\ W_1+jW_2]\Big\} & \subset & \mathbb{HP}^1 \setminus [0,1]
\end{array}$$

$$(W_1+jW_2)(1+ja)^{-1} = z_1+jz_2$$

The fibres $\mathbb{CP}^1$ of π parametrize $SO(4)/U(2)$.

Conclusion. An OCS J on $\Omega \subset \mathbb{R}^4$ can be regarded as a "holomorphic" section $\Omega \to \mathbb{CP}^3$, or rather a map $\Omega \to \mathbb{CP}^3$ whose differential is complex linear relative to the endomorphism J induced on Ω.

Liouville theorems revisited

Let J be an OCS on an open set $\Omega = \mathbb{R}^4 \setminus K$.

$\boxed{1}$ If $\mathcal{H}^1(K) = 0$ then J arises from some plane $\mathbb{CP}^2 \subset \mathbb{CP}^3$.

E.g. J_0 is defined by $a=0$, and J_1 by $W_1=0$.

Sketch proof: $J(\Omega)$ is analytic in $\mathbb{CP}^3 \setminus \pi^{-1}(K)$. Key point is to deduce that $\overline{J(\Omega)}$ is analytic, so algebraic of degree 1.

$\boxed{2}$ If K is a line or circle then J arises from a unique quadric in $\mathbb{CP}^3$.

E.g. J_2 is defined by $W_2 = aW_1$.

Sketch proof: There exists a quadric $\mathscr{Q}$ such that $\pi^{-1}(K)$ disconnects both $J(\Omega)$ and $\mathscr{Q}$. If $\mathscr{C}$ is a remaining component of $J(\Omega)$, we deduce that $\overline{\mathscr{C}}$ is analytic.

Removal of singularities

Hausdorff measure is defined in terms of a cover by sets of increasingly small diameter. If $E_i \subset \mathbb{R}^m$, let $r_i = \frac{1}{2}\text{diam}(E_i)$ and

$$v_i^d = \text{vol}(B^d(r_i)) = \frac{(\pi r_i^2)^{d/2}}{\Gamma(\frac{d}{2}+1)}.$$

$$\mathcal{H}^d(K) = \lim_{\delta\to 0} \inf\left\{ \sum_{i=1}^{\infty} v_i^d : K \subseteq \bigcup_{i=1}^{\infty} E_i,\ r_i < \delta \right\}$$

Shiffman's Theorem (1968). U open in $\mathbb{C}^n$, E closed in U. If $A^{2k} \subset U \setminus E$ is analytic and $\mathcal{H}^{2k-1}(E) = 0$, then $\overline{A} \cap U$ is analytic.

Based on the proof of

Bishop's Theorem (1964). U open in $\mathbb{C}^n$ and B **analytic** in U. If $A^{2k} \subset U \setminus B$ is analytic and $\mathcal{H}^{2k}(\overline{A} \cap B) = 0$ then $\overline{A} \cap U$ is analytic.

A generalization of Remmert-Stein (1953).

Classification of the discriminant locus

Definition. Given a quadric $\mathscr{Q} \subset \mathbb{CP}^3$ and a point $q \in \mathbb{R}^4$, set $D = D_0 \cup D_1$ where

$$q \in D_0 \;\Leftrightarrow\; \pi^{-1}(q) \subset \mathscr{Q}$$
$$q \in D_1 \;\Leftrightarrow\; \pi^{-1}(q) \text{ is 1 point.}$$

Theorem 3 There are three types of non-degenerate quadrics $\mathscr{Q} \subset \mathbb{CP}^3 \to S^4$:

(0) $\mathscr{Q}$ is "real" (meaning invariant by j acting on $\mathbb{C}^4 = \mathbb{H}^2$) and $D = D_0 = S^1$. We get an OCS $\pm J$ on $\mathbb{R}^4 \setminus K$ where $K = D$.

(1) $D = D_1 = S^1 \times S^1$ is a smooth unknotted torus in $\mathbb{R}^4$. We get an OCS's J, J' on $\mathbb{R}^4 \setminus K$ where $K = S^1 \times B^2$ is a solid torus.

(2) D is a torus pinched over two points q_1, q_2 and $D_0 = \{q_1, q_2\}$.

In both (0) and (1), $\pi^{-1}(K) \cap \mathscr{Q} \cong S^1 \times S^2$.

The "real" structure on $\mathbb{CP}^3$

Is induced from multiplication by j:

$$(1, W_1, a, W_2) \ \mapsto \ (-\overline{a}, -\overline{W}_2, 1, \overline{W}_1)$$

A matrix $G \in \mathbb{C}^{4,4}$ belongs to $\mathfrak{gl}(2, \mathbb{H})$ iff

$$G = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}$$

Then $G \in SL(2, \mathbb{H}) \stackrel{2:1}{\to} SO_0(5, 1)$ if $\det G = 1$.

Definition. Let $Q, Q' \in \mathbb{C}^{4,4}$ be symmetric. Then $Q \sim Q'$ if $Q' = \lambda G^\top Q G$ for some $\lambda \in \mathbb{C}^*$ and $G \in SL(2, \mathbb{H})$.

The space of orbits has dimension 20–17 = 3.

Any symmetric $Q \in \mathbb{C}^{4,4}$ equals $P_1 + iP_2$ with $P_\alpha \in \mathfrak{gl}(2, \mathbb{H})$. If $\operatorname{rank} Q = 4$, then $Q \sim I + iP_3$ with $\operatorname{tr} P_3 = 0$.

Conformal canonical form

Given $Q = I + iP_3$, the stabilizer

$$SO^*(4) = SL(2,\mathbb{H}) \cap O(4,\mathbb{C})$$

of I is isomorphic to

$$SL(2,\mathbb{R}) \times_{\mathbb{Z}_2} SU(2) \xrightarrow{2:1} SO(2,1) \times SO(3).$$

P_3 is determined by $X \in \mathbb{R}^{3,3}$, and the space of orbits has dimension $9-6=3$. We use SVD to diagonalize X and Q, and obtain

Theorem 4 Under the action of the group $SO_0(5,1)$ on $\mathbb{CP}^3 \to S^4$, any non-degenerate quadric is represented by the matrix

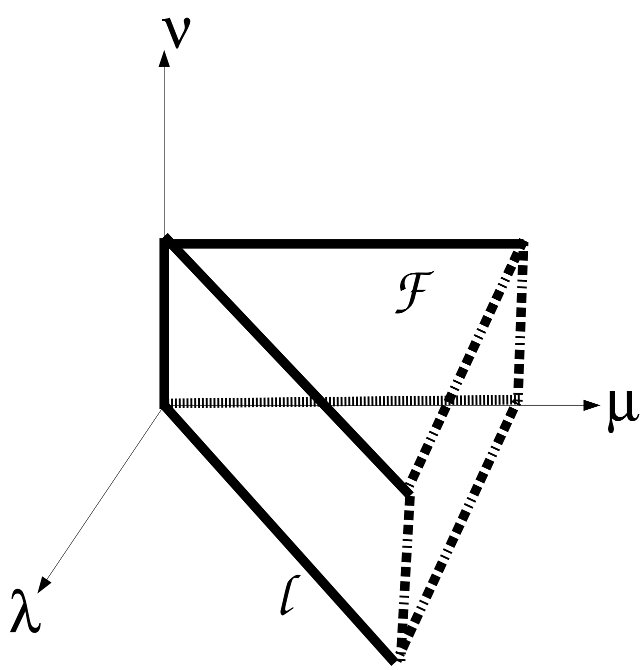
$$\begin{pmatrix} e^{\lambda+i\nu} & 0 & 0 & 0 \\ 0 & e^{\mu-i\nu} & 0 & 0 \\ 0 & 0 & e^{-\lambda+i\nu} & 0 \\ 0 & 0 & 0 & e^{-\mu-i\nu} \end{pmatrix}$$

for some $\lambda, \mu, \nu \in \mathbb{R}$.

A fundamental domain

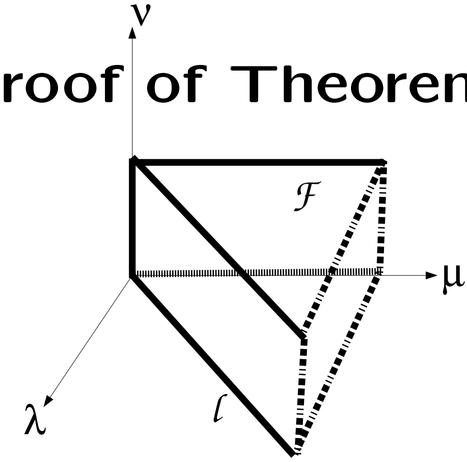
In view of equivalences such as $\nu \leftrightarrow \frac{\pi}{2}-\nu$, we may restrict to

$$\{(\lambda,\mu,\nu) : 0 \leqslant \lambda \leqslant \mu,\ 0 \leqslant \nu < \tfrac{\pi}{2}\}.$$



The origin $\mathbf{0} \sim (0,0,\frac{\pi}{2})$ represents the real quadric and J_2. The pinched case (2) arises from the open half line $\ell = \{(\lambda,\lambda,0) : \lambda > 0\}$. Quadrics lying on the face $\mathcal{F} = \{(0,\mu,\nu)\}$ required special attention.

The proof of Theorem 3



(i) It is straightforward to determine that $D_0 = \emptyset$ outside ℓ, and $\# D_0 = 2$ if $\lambda = \mu > 0$.

(ii) Consider the discriminant

$$\Delta = B^2 - AC : \mathbb{R}^4 \to \mathbb{R}^2;$$

then rank(grad Δ) < 2 only on ℓ.

(iii) $\mathfrak{Im}\,\Delta$ is a smooth 3-sphere in $\mathbb{R}^4$ if $\nu \neq 0$.

(iv) We then use $\chi(D) = 2\chi(S^2) - \chi(\mathscr{Q}) = 0$ to prove that D has no S^2 components, and is in fact connected, at all interior points.

(v) D is then a smooth torus except on ℓ.

Example for case (1): $(0, 0, \frac{\pi}{4})$

The quadratic form equals

$$\begin{aligned} e^{i\nu}(1 + a^2) + e^{-i\nu}({W_1}^2 + {W_2}^2) \\ = Aa^2 + 2Ba + C. \end{aligned}$$

The discriminant $\Delta = B^2 - AC$ equals

$$i - i|\mathbf{z}|^4 + {z_1}^2 + {\overline{z}_1}^2 + {z_2}^2 + {\overline{z}_2}^2$$

in terms of $\mathbf{z} = (z_1, z_2) = (\mathbf{x}, \mathbf{y})$.

The zero set D of Δ is given by

$$\begin{aligned} \mathfrak{Im} &: \quad 1 = |\mathbf{z}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 \\ \mathfrak{Re} &: \quad 0 = |\mathbf{x}|^2 - |\mathbf{y}|^2 \end{aligned}$$

So $|\mathbf{x}| = |\mathbf{y}| = \frac{1}{2}$ and

$$D = D_1 \cong S^1 \times S^1$$

is a smooth Clifford torus.

Example for case (2): $(\lambda, \lambda, 0)$

We can find $G \in SL(2, \mathbb{H})$ such that

$$G^{\top} Q G = \begin{pmatrix} 0 & 0 & 0 & e^{-\lambda} \\ 0 & 0 & -e^{\lambda} & 0 \\ 0 & -e^{\lambda} & 0 & 0 \\ e^{-\lambda} & 0 & 0 & 0 \end{pmatrix}$$

and $\mathscr{Q}$ has equation $2e^{-\lambda} W_2 - 2e^{\lambda} a W_1 = 0$

$D = \{\Delta = 0\}$ is given by

$$2|z_1|^2 + 4|z_2|^2 = e^{2\lambda} {z_1}^2 + e^{-2\lambda} {\overline{z}_1}^2$$

$$\mathfrak{Im} \quad : \quad x_1 y_1 = 0$$

$$\mathfrak{Re} \quad : \quad 2({x_2}^2 + {y_2}^2) = (c-1){x_1}^2$$

If $c = \cosh \lambda > 1$ then D is a cone with vertex at 0 (and $\infty \in S^3$) and $D_0 = \{0, \infty\}$.

Quartics in $\mathbb{CP}^3 \to S^4$

The surface $\mathscr{K}_c$ with equation

$$1 + a^4 + W_1^4 + W_2^4 + 6ca^2 = 0$$

is "real" and non-singular for $c \neq \pm\frac{1}{3}$

Lemma $\mathscr{K}_c$ contains no $\mathbb{CP}^1$ fibres unless $c = -1, 0, 1$, in which cases it has 8.

Generically, $D(\mathscr{K}_c)$ is given by

$$6ABC = 4B^3 - 2\overline{B}A^2 \quad \text{and} \quad A\overline{B}^2 = \overline{A}B^2,$$

where $A = 1 + z_1^4 + z_2^4$, $B = -z_1^3\overline{z}_2 + z_2^3\overline{z}_1$ and $C = z_1^2\overline{z}_2^2 + z_2^2\overline{z}_1^2 + c$.

If $c \notin \{-1, 0, 1\}$, consider $\mathscr{E} = \mathscr{K}_c/\mathbb{Z}_2$ and

$$\mathscr{E} \setminus \pi^{-1}(D) \xrightarrow{2:1} S^4 \setminus D$$

Then $\chi(D) = -8$ and D must be singular.

Higher degrees in general

What are the possible maximal domains of definition $\Omega = \mathbb{R}^4 \setminus K$ for OCS's?

Let $\mathscr{S}$ be an irreducible surface of $\mathbb{CP}^3$ of degree $d \geqslant 2$ with discriminant locus D. If $p \in \pi^{-1}(D) \in \mathscr{S}$ then $\text{rank}(d\pi_p) = 2$ and it follows that $\dim_{\mathcal{H}} D \leqslant 2$.

Theorem 5 $\mathscr{S}$ contains the graph of a single-valued OCS J on $S^4 \setminus K$ only if $K \supseteq D$ and $\mathscr{S} \setminus \pi^{-1}(K)$ is disconnected. Moreover, $\mathcal{H}^3(K) \neq 0$ unless $\mathscr{S}$ is a real quadric.

If $d > 2$, D_0 consists of finitely many points. If $\mathscr{S}$ is j-invariant then $\#D_0 \leqslant d^2$, though better estimates will be possible.