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Preface

Interest in special classes of Riemannian metrics stems from several fronts, but there are two particular reasons for picking out metrics distinguished by a holonomy reduction. First, these metrics frequently satisfy the vacuum Einstein equations, and are consequently the subject of relentless study by physicists. Second, the theory surrounding a particular holonomy group or groups has often turned out to be extremely rich, and worthy of consideration as a separate topic, endowed with its own characteristic techniques. The foremost example is the theory of Kähler manifolds, based on the unitary group, which blends together complex and Riemannian geometry. Examples arising from various other groups at first seemed elusive, but are now known to exist in abundance.

These notes represent an expanded version of a 1986 lecture course, which was motivated in part by a seminar that Robert Bryant had presented on the local existence of metrics with exceptional holonomy groups. His visit to Oxford on that occasion signalled the beginning of a joint search for explicit examples of such metrics, some of which it was possible to describe in the course. For my part, the realization of these examples was accelerated by ideas I had encountered in the work of both Alfred Gray and Stefano Marchiafava. I should also like to thank Nigel Hitchin for presenting the problem to me in the first instance, and for the influence he has had on my work.

Many other people have assisted, sometimes unwittingly, with the production of this volume. I am grateful particularly to Simon Donaldson for his constant encouragement and interest, and also to Piotr Kobak, Claude LeBrun, Marco Mamone Capria, Sun Poon and Andrew Swann for useful suggestions.

Simon Salamon

Introduction

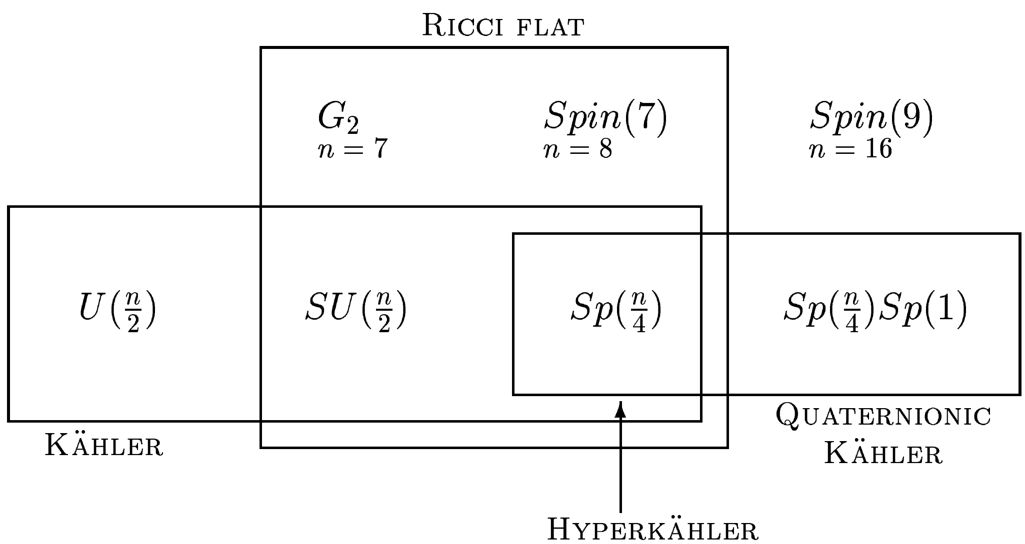
A Riemannian manifold M having n dimensions is modelled at each point on a Euclidean vector space with a standard inner product. The group $O(n)$ of orthogonal transformations that preserve this inner product appears as the "structure group" of the manifold, and all natural operations on M are compatible with the pointwise action of this group. For example, the canonical connection on M allows one to set up parallel transport, which will define an isometry between tangent spaces at two points x, y, given a path between them. Dependence on the path is measured by taking x and y coincident, and parallel transport along all loops at x generates the so-called holonomy group H, which is then a subgroup of $O(n)$.

The action of the holonomy group H provides a way of assessing how an object defined on M varies from point to point. Of particular significance is the way that H acts on the curvature tensor R, which itself detects the infinitesimal effect of parallel transport, and it is not surprising that there is an intimate link between curvature and holonomy. At a given point, the curvature tensor can be thought of as a linear map that assigns to any two vectors x, y a skew-symmetric transformation R_{xy}. Generically, these transformations would be expected to generate the Lie algebra of all skew-symmetric endomorphisms, which implies (if M is oriented) that H equals $SO(n)$. If the holonomy group is a proper subgroup of $SO(n)$, then M is "special" in some sense; for example if H acts reducibly on $\mathbb{R}^n$, then the metric is locally a product. One of the main purposes of these notes to elaborate the special features in detail for groups H that act irreducibly on $\mathbb{R}^n$.

Symmetric spaces are characterized by the invariance of the curvature tensor R under parallel translation, which corresponds to the algebraic statement that H acts trivially on R. The classification of symmetric spaces was accomplished by É. Cartan in the 1920's, and involves a remarkable correspondence with simple Lie algebras. It also provides quite a large list of different holonomy groups, but the case of metrics which are not symmetric was not tackled until much later. In 1955, Berger proved the following result.

Theorem Let M be an oriented simply-connected n-dimensional Riemannian manifold which is neither locally a product nor symmetric. Then H must equal one of $SO(n)$, $U(\frac{n}{2})$, $SU(\frac{n}{2})$, $Sp(\frac{n}{4})Sp(1)$, $Sp(\frac{n}{4})$, G_2, $Spin(7)$ or $Spin(9)$.

The list is not only one of groups, but also one of representations, the action of each group on the tangent space being completely specified. Consequently each of the groups apart from $SO(n)$ gives rise to a structure with its own geometrical flavour. The structures can be broadly divided into three categories, corresponding to Kähler (and therefore complex) manifolds, quaternionic manifolds (whose real dimension is a multiple of four), and finally manifolds whose Ricci tensor is zero (which therefore cannot also be holonomy groups of symmetric spaces). Strictly speaking, each category is characterized by the inclusion of H in one of the indicated groups, and the intersection of all three defines the class of so-called hyperkähler manifolds, which includes the group $SU(2) = Sp(1)$ when $n = 4$.



There remained the question of which of these groups actually arise from the holonomy of non-symmetric metrics. The answer, namely all except $Spin(9)$, has resulted from subsequent work that has only been completed in the last few years, and that we shall describe presently. The observation by Borel that Berger's list coincided almost exactly with the list of Lie groups that can act transitively on a sphere led Simons to prove this fact directly in 1962, thereby giving an independent proof of the above theorem. The two Bianchi identities play fundamental and sometimes mysterious roles in Riemannian geometry, and whereas Berger had exploited consequences of both of them, Simons succeeded in proving an algebraic version of the theorem by exploiting the full power of just the first identity. A new proof of

Berger's theorem that avoids some of the classification issues has recently been found by Bryant.

By far the greatest impetus to the holonomy problem since Berger's theorem has come from Yau's resolution of the Calabi conjecture. This effectively converts the study of compact manifolds with holonomy group $SU(m)$ or $Sp(k)$ to the study of Kählerian manifolds with zero first Chern class. Here "Kählerian" means admitting some (possibly uninteresting) Kähler metric, the point being that the existence is then guaranteed of another Kähler metric with the reduced holonomy. Projective geometry then provides an abundant source of manifolds which admit a Riemannian metric with zero Ricci tensor, but there remains the problem of describing these metrics in a more explicit way. Calabi's construction of Ricci-flat Kähler metrics on the total space of vector bundles not only provides some sort of first approximation to the Yau metrics, but can in some sense be used in collaboration with them to generate new compact examples.

Metrics whose holonomy group is contained in $Sp(k)$ are called hyperkähler because they are simultaneously Kähler with respect to a family of complex structures. They constitute the most important examples of a much larger class of quaternionic manifolds, but are usefully studied from the viewpoint of symplectic geometry. Although there seems to be no meaningful theory of quaternionic submanifolds, the discovery by Hitchin, Karlhede, Lindström and Roček that hyperkähler manifolds are amenable to being quotiented has had far-reaching implications. One of its corollaries is the existence of metrics with holonomy equal to $Sp(k)$ on a large class of moduli spaces of solutions to the Yang-Mills equations.

Finding compact manifolds with holonomy group equal to $Sp(k)$ for $k \geqslant 2$ proved to be more delicate, but examples were eventually forthcoming despite belief at one stage that none existed. The work on hyperkähler manifolds has also led to a greater understanding of metrics with holonomy group equal to $Sp(k)Sp(1)$. Although this group arises as the isotropy subgroup of a symmetric space, and is not in general the holonomy group of a complex manifold, it is a little surprising that the enlargement from $Sp(k)$ to $Sp(k)Sp(1)$ does not lead to greater flexibility. However, there do appear to be intimate links between metrics with these two holonomy groups.

The fact that any Riemannian manifold with holonomy group $Spin(9)$ is locally isometric to the Cayley projective plane or its dual is mainly the result of an algebraic calculation, first carried out by Alekseevskii. The local existence of metrics with holonomy group equal to G_2 and $Spin(7)$ was established in the 1980's by Bryant using, appropriately enough, techniques which Cartan had developed more than fifty years previously. Increased confidence of finding such metrics has since led to the discovery of complete examples on total spaces of vector bundles over manifolds of dimension three and four. Here, we have chosen to describe these examples in a wider context, and this we do at the expense of treating the abstract existence theory. The metrics themselves are remarkably simple to describe; although only by following the steps in their construction can one fully appreciate the reason for the holonomy reduction. A more direct and thorough treatment appears in the joint paper of Bryant and the author. Bearing in mind the history concerning other holonomy groups, and the one-time pessimism about the existence of any metrics with G_2 or $Spin(7)$ holonomy, it seems not unreasonable to suppose that compact examples do exist.

As we have made clear, our ultimate concern is more with individual classes of Riemannian metrics and manifolds, than with the general theory of holonomy groups. Nevertheless, two basic concepts of differential geometry pervade the majority of chapters, namely torsion and curvature. An equivalent way of asserting the existence of a Riemannian manifold M with a given holonomy group H is by means of a torsion-free connection preserving a reduction of structure to H. This explains why an understanding of torsion is essential for a systematic treatment of different structure groups, and at several key points we implicitly make use of the related notion of weak holonomy group, introduced by Gray. The relevance of curvature has already been indicated, but its full appreciation necessitates a working knowledge of the representations of compact Lie groups. This is something we have attempted to build up from scratch, which explains an algebraic bias and the inclusion of chapter 6 tackling the practicalities of decomposing tensor products.

We shall not dwell on the contents of individual chapters here, since each has its own summary. They are arranged as follows. The first two are preparatory, and include relevant definitions of the holonomy group, but we have not duplicated proofs of standard theorems which already appear in a number of good references. At this juncture, the reader may care to read the excellent survey on holonomy groups in Besse's book on Einstein Manifolds. Chapters 3,4,5 begin dealing with various applications and contain standard results interspersed with less familiar material. An outline of the classification theorem for irreducible manifolds based on Simons's approach is included for completeness in chapter 10, so that recent research is the subject of the remaining chapters 7,8,9,11,12. Of these, the seventh is the most crucial, since it promotes techniques that are employed frequently thereafter.

The notes still follow the broad outline of a lecture course in which chapter was originally responsible for about one lecture, and as a consequence do not aspire to give an exhaustive survey of the subject. For the sake of simplicity, we restrict ourselves entirely to consideration of positive definite metrics, although there are many unresolved problems in the pseudo-Riemannian case. Only the tip of the iceberg forming the theory of Ricci-flat Kähler manifolds is revealed, and (except in chapter 8) there is little discussion of the interesting topological restrictions that accompany a holonomy reduction, and which are likely to point the way to further examples. On the other hand, the notes do cover a range of topics in Riemannian geometry, in which Lie groups play a fundamental role.

1 Manifolds and Structure Groups

The aim of this chapter is to summarize basic techniques concerning the differential geometry associated to a smooth manifold M. Because of the important role that Lie groups will play, we choose to define many concepts in terms of the principal frame bundle LM of M, at least in the first instance. This ensures a coordinate-free approach, and encourages one to treat total spaces of bundles as manifolds in their own right. A subgroup H of $GL(n,\mathbb{R})$ determines its own brand of geometry, which is implemented by the choice of a subbundle P of LM with fibre H; the Riemannian case corresponds to the orthogonal group $H = O(n)$.

Linked with the definition of such an H-structure P, there is the notion of an H-connection which permits one to differentiate tensors arising from representations of H. This formalism will be invaluable in the sequel, when H will often stand for a holonomy group which equals $O(n)$ or one of its subgroups. In general, the existence of connections compensates for the absence of canonical coordinate systems, and in some sense the compatibility of a particular connection with coordinates is measured by its torsion. This will be quantified at the end of the chapter, even though thereafter we shall be working exclusively with connections whose torsion is zero.

The first few pages are designed to establish some basic notation, which will appear frequently.

Preliminaries

Let M be a smooth paracompact manifold of dimension n. Thus, M has an atlas consisting of charts (U_α, ϕ_α), where $\{U_\alpha\}$ is an open covering of M, and the maps $\phi_\alpha\colon U_\alpha \to \mathbb{R}^n$ are homeomorphisms onto open sets of Euclidean space with $\phi_\beta \circ \phi_\alpha^{-1}$ infinitely differentiable wherever it is defined. The smooth structure of M is then specified by assigning to *every* open set U of M the space $C^\infty U$ of functions $f\colon U \to \mathbf{R}$ for which $f \circ \phi_\alpha^{-1}$ is always infinitely differentiable.

At $m \in M$, there is a well-defined subspace A_m of $C^\infty M$ consisting of functions which are constant to first order at m, and the cotangent space at m may be defined as the quotient

$$T_m^*M = \frac{C^\infty M}{A_m}.$$

The projection $C^\infty M \to T_m^*M$ maps a function f to its differential df_m. Of course, M may be replaced by any open set containing m, and if m lies in the domain U_α of a chart with coordinates $\phi_\alpha = (x^1, \ldots, x^n)$, then we may write $A_m = \{f \in C^\infty M : \frac{\partial f}{\partial x^i} = 0, \forall i\}$. The tangent space T_mM is the space of $\mathbb{R}$-linear maps $X\colon C^\infty M \to \mathbb{R}$ annihilating A_m, and has a basis consisting of the operators $\frac{\partial}{\partial x^i}$.

A vector field is, intuitively, a smooth assignment to each $m \in M$ of an element of T_mM. Equivalently, it is an $\mathbb{R}$-linear map $C^\infty M \to C^\infty M$ satisfying the derivation property

$$X(fg) = (Xf)g + f(Xg), \quad f, g \in C^\infty M,$$

since $(f - f(m))(g - g(m)) \in A_m$. The commutator $[X,Y] = XY - YX$ then gives the set of vector fields $\mathfrak{X}M$ the structure of a Lie algebra. In general, a Lie algebra is a vector space $\mathfrak{g}$ with an alternating bilinear map or bracket $[\,,\,]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \tag{1.1}$$

For each $k \geqslant 1$, the bracket extends to

$$b\colon \textstyle\bigwedge^{k+1}\mathfrak{g} \hookrightarrow \bigwedge^2\mathfrak{g} \otimes \bigwedge^{k-1}\mathfrak{g} \xrightarrow{[\,,\,]\otimes 1} \mathfrak{g} \otimes \bigwedge^{k-1}\mathfrak{g} \longrightarrow \bigwedge^k\mathfrak{g}, \tag{1.2}$$

in which the first map is an inclusion, and the last is anti-symmetrization. The Jacobi identity is equivalent to the assertion that $b^2 = 0$.

If we treat $\mathfrak{X}M$ as a module over the ring $C^\infty M$, then $\bigwedge^k\mathfrak{X}M$ is dual to the set Ω^kM of differential forms of degree k. A element σ of Ω^kM is a smooth assignment $m \mapsto \sigma_m \in \bigwedge^k T_m^*M$. The de Rham complex

$$0 \to C^\infty M = \Omega^0 M \xrightarrow{d} \Omega^1 M \xrightarrow{d} \Omega^2 M \longrightarrow \cdots \longrightarrow \Omega^n M \to 0$$

is formed from the exterior derivative $d\colon \Omega^k M \to \Omega^{k+1}M$, which is an anti-derivation extending the differential on functions. The de Rham cohomology groups of M are defined by

$$H^k(M,\mathbb{R}) = \frac{\ker(d\colon \Omega^k M \to \Omega^{k+1}M)}{d(\Omega^{k-1}M)},$$

and their dimensions equal the *Betti numbers* b_k of M.

The only formula relating the definition of d to Lie brackets that is likely to concern us is

$$2d\sigma(X,Y) = X(\sigma Y) - Y(\sigma X) - \sigma[X,Y]. \tag{1.3}$$

for a 1-form σ. The initial "2" is optional; its insertion here results from our convention of identifying the exterior product $v \wedge w$ with $\frac{1}{2}(v \otimes w - w \otimes v)$. Given a smooth map $F\colon M \to N$ between manifolds with $f(m) = n$, we write

$$F_*\colon T_m M \to T_n N, \qquad F^*\colon T_n^* N \to T_m^* M,$$

for the induced linear maps, the second of which extends to a cochain mapping of the de Rham complex.

The integral curves or flow lines associated to a single vector field X determine a local one-parameter group of diffeomorphisms of M. Consider instead a subspace $\mathfrak{X}'$ of $\mathfrak{X}M$ defining a *distribution* of subspaces $T_m' = \{X|_m \in T_m M\colon X \in \mathfrak{X}'\}$ of constant dimension $p \geqslant 1$. The *Frobenius theorem* asserts that the T_m' are tangent to a submanifold of M if and only if $\mathfrak{X}'$ is a Lie algebra. By (1.3), this is equivalent to the assertion that the ideal of differential forms annihilating $\mathfrak{X}'$ is closed under exterior differentiation.

A Lie group can be defined as a group H which has the structure of a smooth n-dimensional manifold for which the multiplication $H \times H \to H$ is a smooth map. The tangent space $T_e H$ to H at the identity can be identified with an n-dimensional subalgebra $\mathfrak{h}$ of $\mathfrak{X}H$, equal to the space of *left-invariant* vector fields. Each such vector field X generates a one-parameter subgroup $h_t = \exp(tX)$ of H, that defines the *exponential mapping* $\exp\colon \mathfrak{h} \to H$. More generally, any subalgebra $\mathfrak{h}'$ of $\mathfrak{h}$ is tangent to a unique connected Lie subgroup H' of H. In the sequel, we shall also use freely the bijective correspondence between connected, simply-connected Lie groups and finite-dimensional Lie algebras [Ch].

Let H be a Lie group. If $X_1, \ldots, X_n$ form a basis of $T_e H$, with dual 1-forms $\omega^1, \ldots, \omega^n$, then $\omega = \sum\limits_{i=1}^{n} \omega^i \otimes X_i$ may be treated as a canonical 1-form on H with values in the Lie algebra $\mathfrak{h}$. Because of (1.3), it satisfies the *Maurer-Cartan equation*

$$d\omega = -\tfrac{1}{2} \sum_{j,k=1}^{n} (\omega^j \wedge \omega^k) \otimes [X_j, X_k] = -\tfrac{1}{2}[\omega, \omega]. \tag{1.4}$$

The brackets on the right-hand side indicate wedging the 1-forms together, and performing the Lie bracket at the same time, in accordance with the inclusion of $\bigwedge^2(\mathbb{R}^n)^*\otimes\bigwedge^2\mathfrak{h}$ in the symmetric product $\bigodot^2((\mathbb{R}^n)^*\otimes\mathfrak{h})$.

The Maurer-Cartan form ω characterizes the local geometry of H, in the following manner. Let M be a manifold equipped with a 1-form ω^M with values in the Lie algebra $\mathfrak{h}$, and satisfying (1.4). Then each point of M has a neighbourhood U admitting a diffeomorphism $f\colon U \to f(U) \subset H$ with $\omega^M = f^*\omega$. For more details see, for example, [S].

The principal frame bundle

When the tangent space T_mM has no preferred basis, a remedy is to consider all bases simultaneously. Any such basis or *linear frame* can conveniently be described by an isomorphism $p\colon \mathbb{R}^n \to T_mM$. Their union as m varies is the *principal frame bundle* LM, and is naturally a smooth manifold with a projection $\pi\colon LM \to M$, and the following properties:

(i) The group $GL(n,\mathbb{R})$ of non-singular transformations of $\mathbb{R}^n$ acts freely on LM on the right with the quotient $LM/GL(n,\mathbb{R})$ isomorphic to M. If $p\colon \mathbb{R}^n \to T_mM$, and $g \in GL(n,\mathbb{R})$, we write

$$R_g(p) = pg = p\circ g.$$

(ii) A "moving frame", consisting of a set $\{X_1,\ldots,X_n\}$ of smooth, everywhere linearly independent, vectors fields on an open set U of M, determines a section $s\colon U \to LM$, $\pi\circ s = \mathbf{1}$. Let $\Gamma(U,LM)$ denote the set of such sections; then any $s \in \Gamma(U,LM)$ defines a local trivialization

$$\pi^{-1}U \cong U \times GL(n,\mathbb{R}).$$

(iii) There are special elements $s = \{X_1,\ldots,X_n\}$ of $\Gamma(U,LM)$ which we shall call *integrable sections*, characterized by the property that all Lie brackets $[X_i,X_j]$ vanish. In these circumstances, if U is sufficiently small, it is the domain of a chart $(x^1,\ldots,x^n)$ which induces s so that $X_i = \frac{\partial}{\partial x^i}$.

It is the last property that reminds us that LM has features not enjoyed by an arbitrary principal bundle on M. Indeed, there exists on LM a canonical or "soldering" 1-form θ with values in $\mathbb{R}^n$, defined at $p \in LM$ by

$$\theta(Y) = p^{-1}(\pi_*Y), \quad Y \in T_p(LM). \tag{1.5}$$

If s is the integrable section $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$, then $s^*\theta$ is the dual basis $\{dx^1, \ldots, dx^n\}$ of 1-forms. Thus integrable sections are characterized by the condition $0 = d(s^*\theta) = s^*(d\theta)$, which leads to the following pointwise statement.

1.1 Lemma *A subspace D of $T_p(LM)$ is tangent to a integrable section if and only if (i) it is horizontal in the sense that the restriction $\pi_*|_D$ is an isomorphism, and (ii) $d\theta|_{\Lambda^2 D} = 0$.*

Such a subspace D represents the 2-jet of a local diffeomorphism from $\mathbb{R}^n$ into M, or an element of the principal bundle of "2-frames" on M, to use the formalism exploited by Kobayashi and others [K$_3$].

If H is a subgroup of $GL(n,\mathbb{R})$, an H-*structure* on M is a principal subbundle P of LM with group H. This means that, given P, there exists an open covering $\{U_\alpha\}$ of M and local sections $s_\alpha: U_\alpha \to LM$ such that

$$P_m = \pi^{-1}(m) \cap P = \{s_\alpha(m)h : h \in H\}, \quad m \in U_\alpha,$$

and we write $s_\alpha \in \Gamma(U_\alpha, P)$. The *transition functions*

$$h_{\alpha\beta}: U_\alpha \cap U_\beta \to H$$

of P are defined by $s_\beta = s_\alpha h_{\alpha\beta}$. The fact that they have values in H rather than $GL(n,\mathbb{R})$ expresses the fact that the structure group of LM has been *reduced* to the subgroup H, and an H-structure P should be regarded as a set of "distinguished frames". To the extent that *we* shall require it, an H-structure really just provides a useful way of describing and unifying concepts which are already familiar and commonly used.

Consider the orthogonal group $O(n)$ of linear transformations of $\mathbb{R}^n$ preserving a positive definite inner product. Then an $O(n)$-structure is none other than a Riemannian metric, or a smoothly varying inner product g on each tangent space T_mM. The standard inner product on $\mathbb{R}^n$ (thought of as a space of column vectors) is transferred to T_mM by setting

$$g(X,Y) = (p^{-1}X)^t(p^{-1}Y), \quad X,Y \in T_mM, \tag{1.6}$$

for any $p \in P_m$ (the choice is clearly immaterial). Elements of P are precisely the isometries $\mathbb{R}^n \to T_mM$ or, in terms of bases, the orthonormal frames.

If $GL^+(n,\mathbb{R})$ denotes the identity component of $GL(n,\mathbb{R})$, consisting of matrices with positive determinant, then a $GL^+(n,\mathbb{R})$-structure Q defines an *orientation* on

M, and its elements are those frames compatible with the orientation. Superimposing the two examples determines the $SO(n)$-structure $P \cap Q$ of oriented orthonormal frames.

An H-structure P is said to be *integrable* if it admits integrable sections, so that given $m \in M$, there exists $s \in \Gamma(U, P)$, $m \in U$, with $s^* d\theta = 0$. The bundles P and Q above illustrate extreme behaviour; Q is always integrable, whereas P is integrable if and only if in a neighbourhood of any point, there exist local coordinates with $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ orthonormal. In the latter case the Riemannian metric has locally the standard form $\sum_{i=1}^{n} dx^i \otimes dx^i$, the flat situation considered further in **2.7**.

A *linear representation* of a Lie group H is determined by a continuous homomorphism $H \to \operatorname{Aut} V$, where V is a vector space which we are apt to refer to as an H-*module.* If M is a smooth manifold with an H-structure P, then a function f on P with values in V, which is *equivariant* in the sense that

$$f(ph) = h^{-1}(f(p)), \quad h \in H, \tag{1.7}$$

defines what is called a *tensor* on M. As p ranges over P, the pair $(p, f(p))$ determines a section of the vector bundle $VM = P \times_H V$ over M, obtained by taking the quotient of $P \times V$ by the right action of H given by $(p, v)h = (ph, h^{-1}v)$. This vector bundle is called the *bundle associated to P with fibre V.*

For example, if V is an H-invariant subspace of $\bigwedge^k(\mathbb{R}^n)^*$, then f extends to the whole of the frame bundle LM, and defines a k-form lying in a distinguished subbundle VM of $\bigwedge^k T^*M$. More generally, we may replace V by any set $\mathcal{V}$ upon which H acts on the left (that is, $ev = v$ and $(gh)v = g(hv)$ for all $v \in \mathcal{V}$ and $g, h \in H$), to form the bundle $P \times_H \mathcal{V}$. For example the coset space $\mathcal{V} = GL(n, \mathbb{R})/H$ parametrizes reductions of $GL(n, \mathbb{R})$ to the subgroup H, and an H-structure defines in a tautologous fashion a section of the associated bundle. If $\mathcal{V}$ is contractible, such a section will always exist, since by assumption M admits partitions of unity. Applying the argument for $H = O(n)$ establishes the existence of a Riemannian metric.

Connections and covariant differentiation

Let us consider the problem of differentiating a tensor, described by an equivariant function f on the principal frame bundle LM, with values in a fixed vector space V, acted on by $GL(n, \mathbb{R})$. At a point p of LM, the effect of the differential df on vectors tangent to the fibres is clearly determined by $f(p)$. Let $C_p = \ker(\pi_*)$ denote the *vertical space* at p, and $\mathfrak{gl}(n,\mathbb{R}) \cong T_e GL(n,\mathbb{R})$ the Lie algebra of all linear maps $\mathbb{R}^n \to \mathbb{R}^n$. Then there is a natural isomorphism

$$\phi \colon C_p \xrightarrow{\cong} \mathfrak{gl}(n,\mathbb{R}),$$

characterized by the property that

$$df(X) = (Xf)(p) = -\phi(X)f(p), \quad X \in C_p. \tag{1.8}$$

To evaluate the differential df in directions transverse to the fibres, one introduces a *connection*, which consists of a smooth equivariant distribution

$$D_p \subset T_p LM, \qquad D_{pg} = (R_g)_* D_p$$

of horizontal subspaces in LM. This gives rise to an equivariant splitting

$$T_p LM = C_p \oplus D_p, \tag{1.9}$$

and ϕ may be extended to a 1-form on LM with values in $\mathfrak{gl}(n,\mathbb{R})$ by declaring that its restriction to D_p be zero. From (1.5), (1.7) and (1.8), we have

$$(R_g)^*\theta = g^{-1}\theta, \quad (R_g)^* df = g^{-1} df, \quad (R_g)^*\phi = g^{-1}\phi,$$

where $GL(n,\mathbb{R})$ acts on $\mathfrak{gl}(n,\mathbb{R})$ by the adjoint representation or, in other words, conjugation. The V-valued 1-form $df + \phi f$ then satisfies the same equivariance law, and represents the horizontal component of df. Hence, there exists an equivariant function ∇f with values in the tensor product $(\mathbb{R}^n)^* \otimes V$ such that

$$\theta \nabla f = df + \phi f, \tag{1.10}$$

where juxtaposition with θ denotes the obvious contraction that converts elements of $(\mathbb{R}^n)^*$ to 1-forms on LM. More informally, the covariant derivative is given by "$\nabla = d + \phi$".

The 1-form $\phi + \theta$ on LM has values in $\mathfrak{gl}(n,\mathbb{R}) \oplus \mathbb{R}^n$, which may be identified with the Lie algebra formed by extending the bracket of $\mathfrak{gl}(n,\mathbb{R})$ as follows:

$$\begin{aligned} &[A, x] = -[x, A] = Ax, \\ &[x, y] = 0, \qquad A \in \mathfrak{gl}(n,\mathbb{R}), \quad x, y \in \mathbb{R}^n. \end{aligned} \tag{1.11}$$

This is the Lie algebra of the affine group, which in the present circumstances is best thought of as the frame bundle $L\mathbb{R}^n$. Thus $\phi+\theta$ defines an "absolute parallelism" or $\{e\}$-structure on LM, and from (1.4) its failure to conform with the corresponding Maurer-Cartan form is measured by

$$d(\phi+\theta)+\tfrac{1}{2}[\phi+\theta,\phi+\theta] \;=\; \Phi+\Theta,$$

where

$$\begin{aligned}\Phi &= d\phi+\tfrac{1}{2}[\phi,\phi],\\ \Theta &= d\theta+[\phi,\theta],\end{aligned}\tag{1.12}$$

and Lie brackets are now carried out simultaneously with wedging together of 1-forms, in the same vein as (1.4). The exterior derivative of (1.10) yields the so-called *Ricci identity*:

1.2 Lemma $\Phi f-\Theta\nabla f+(\theta\wedge\theta)\nabla^2 f=0.$

Suitable choices of f show that Φ and Θ are "horizontally-valued" 2-forms on LM, being equal to zero upon insertion of a vertical vector. In analogy to (1.10), they are the contractions with $\theta\wedge\theta$ of equivariant functions $R(p)$, $T(p)$, known as the *curvature* and *torsion tensors*, with values in

$$\textstyle\bigwedge^2(\mathbb{R}^n)^*\otimes\mathfrak{gl}(n,\mathbb{R}) \;\cong\; \mathrm{Hom}\,(\bigwedge^2\mathbb{R}^n,\mathfrak{gl}(n,\mathbb{R}))$$

$$\textstyle\bigwedge^2(\mathbb{R}^n)^*\otimes\mathbb{R}^n \;\cong\; \mathrm{Hom}\,(\bigwedge^2\mathbb{R}^n,\mathbb{R}^n)$$

respectively. These functions may be defined more simply as the compositions

$$\begin{aligned}R(p)\colon \textstyle\bigwedge^2\mathbb{R}^n &\xrightarrow{\cong} \textstyle\bigwedge^2 D_p \xrightarrow{\;d\phi\;} \mathfrak{gl}(n,\mathbb{R}),\\ T(p)\colon \textstyle\bigwedge^2\mathbb{R}^n &\xrightarrow{\cong} \textstyle\bigwedge^2 D_p \xrightarrow{\;d\theta\;} \mathbb{R}^n,\end{aligned}\tag{1.13}$$

in which the initial isomorphisms are induced by θ. The connection is said to be *torsion-free* if $T(p)$ is everywhere zero. This means that the restriction of $d\theta$ to $\bigwedge^2 D_p$ vanishes for all $p\in P$, and from **1.1**, each D_p is tangent to an integrable section. The interpretation of $R(p)$ is even clearer; by the Frobenius theorem, it is the obstruction to the integrability of the distribution of horizontal spaces D_p.

Given an H-structure $P \subset LM$, one may consider connections which are compatible with P in the sense that $D_p \subset T_pP$ for every $p \in P$. In this case, the connection *reduces* to one on the principal bundle P, and one loses no information by restricting attention to P. Such H-*connections* are characterized by the fact that the pullback of the 1-form ϕ to P takes values in the Lie algebra $\mathfrak{h}$ of H. If H equals the stabilizer of some element η in a $GL(n,\mathbb{R})$-module V, that element extends to an equivariant function on LM, constant on P. In these circumstances,

1.3 Lemma *A connection reduces to P if and only if η is covariant constant, that is, $\nabla\eta = 0$.*

The reader is welcome to interpret tensors in terms of sections of appropriate bundles on M, as explained after (1.7). Moreover, one can always presume to have an H-connection, by taking $H = GL(n,\mathbb{R})$ and $P = LM$ to cover the general case. The curvature tensor of an H-connection becomes a section of the associated vector bundle

$$P \times_H (\textstyle\bigwedge^2(\mathbb{R}^n)^* \otimes \mathfrak{h}) \cong \textstyle\bigwedge^2 T^*M \otimes \mathfrak{h}M \tag{1.14}$$

over M, where $\mathfrak{h}M = P \times_H \mathfrak{h}$ is the so-called *adjoint bundle*. However, the principal bundle formalism does provide a clear picture of what effect the group has on things. For example, we will be particularly interested in the situation when the curvature $R(p)$ is constant on the fibres of an H-structure P; in this case it is *invariant* rather than just equivariant.

The covariant derivative (1.10) may be regarded as a differential operator acting on sections of associated vector bundles, enjoying the Koszul property

$$\nabla(\lambda f) = \lambda\nabla f + d\lambda \otimes f, \quad \lambda \in C^\infty M. \tag{1.15}$$

For example, if $s = \{X_1, \ldots, X_n\}$ is a section of P over an open set U of M, then one writes

$$\nabla X_j = \sum_k (s^*\phi)_j^k \otimes X_k, \qquad \nabla_Z X_j = \sum_k (\phi(s_* Z))_j^k X_k,$$

where $Z \in T_mM$ and $m \in U$. When s is a integrable section with $X_j = \frac{\partial}{\partial x^j}$, this information is carried by the *Christoffel symbols* $\Gamma_{ij}^k = \phi(s_* \frac{\partial}{\partial x^i})_j^k$, so that

$$\nabla_{\partial/\partial x^i} \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

The 1-form $s^*\phi$ measures the non-horizontality of s, and the matrix entries $(s^*\phi)_j^k$ are the *connection 1-forms* relative to the chosen basis or "gauge". A smooth function $g\colon U \to H$ gives rise to a new gauge sg and new 1-form

$$(sg)^*\phi \;=\; \mathrm{Ad}(g^{-1})s^*\phi + g^{-1}dg,$$

with a transformation law analogous to (1.15).

Measuring torsion

Consider first the special case of **1.2** in which f is (the pullback of) an ordinary $\mathbb{R}$-valued function, so that $\Theta\nabla f = (\theta\wedge\theta)\nabla^2 f$ and $\theta\nabla f = df$. Replacing ∇f by an arbitrary equivariant $(\mathbb{R}^n)^*$-valued function α on LM gives

1.4 Lemma $\Theta\alpha = (\theta\wedge\theta)\nabla\alpha - d(\theta\alpha)$.

Here θ should be regarded as performing a tautological role, and α is really a 1-form on M. As such, the torsion of a connection simply measures the difference between the skew-symmetric component of $\nabla\alpha$, and $d\alpha$. If the torsion is zero, the exterior derivative factors through the covariant derivative; it is easy to see that this is also true for forms of arbitrary degree.

An equivalent formalism can be given in local coordinates $x^1,\dots,x^n$ on M. From **1.4**, the components of the torsion tensor T are given by

$$\begin{aligned} T(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})(dx^k) &= \langle\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j},\nabla dx^k\rangle \\ &= \tfrac12(\Gamma_{ij}^k-\Gamma_{ji}^k). \end{aligned} \tag{1.16}$$

In terms of arbitrary vector fields $X=\sum\limits_i a_i\frac{\partial}{\partial x^i}$, $Y=\sum\limits_j b_j\frac{\partial}{\partial x^j}$, we obtain

$$\begin{aligned} \nabla_X Y - \nabla_Y X &= \sum_{i,j}\Big(a_i\frac{\partial b_j}{\partial x^i}\frac{\partial}{\partial x^j} - b_j\frac{\partial a_i}{\partial x^j}\frac{\partial}{\partial x^i}\Big) + \sum_{i,j,k} a_i b_j(\Gamma_{ij}^k-\Gamma_{ji}^k)\frac{\partial}{\partial x^k} \\ &= [X,Y] + 2T(X,Y). \end{aligned} \tag{1.17}$$

The difference of two connections is a tensorial object. For let ϕ',ϕ be connection forms on an H-structure P. In view of (1.8), their difference $\phi'-\phi$ annihilates vertical vectors, and therefore equals the contraction of θ with an equivariant function

$$\xi(p) \;\in\; \mathrm{Hom}\,(\mathbb{R}^n,\mathfrak{h}) \cong (\mathbb{R}^n)^*\otimes\mathfrak{h}, \tag{1.18}$$

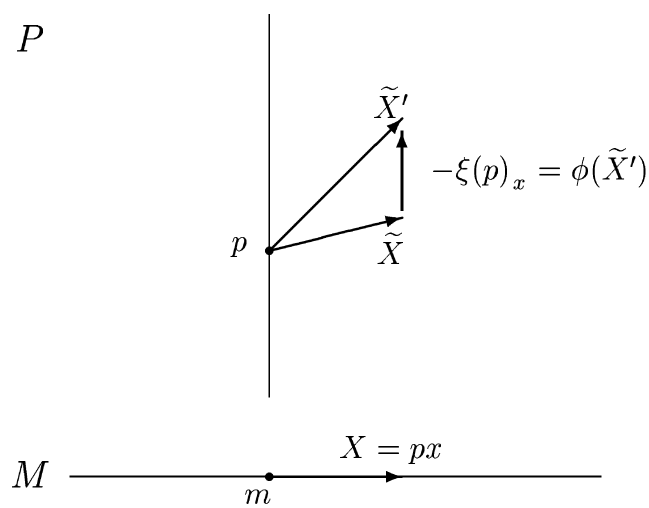
which itself determines a section of the associated vector bundle Hom $(TM, \mathfrak{h}M)$ over the manifold M.

Another interpretation of the last fact starts from the homomorphism $\pi_*: TP \to \pi^{-1}TM$ of vector bundles on P, where $\pi^{-1}TM$ denotes the pulled-back bundle whose fibre at $p \in P$ is by definition $T_{\pi(p)}M$. The kernel of π_* is the bundle of vertical vectors, naturally isomorphic to $P \times \mathfrak{h}$. Quotienting by the action of H yields a short exact sequence

$$0 \to \mathfrak{h}M \longrightarrow \frac{TP}{H} \overset{\longleftarrow}{\longrightarrow} TM \to 0 \tag{1.19}$$

of vector bundles over M. Then an H-connection is exactly the same as a splitting of (1.19); this interpretation was exploited by Atiyah [A_1] in the holomorphic category, where there are obstructions to the existence of connections.

1.5 Figure Difference of two connections



The indicated splitting of (1.19) assigns to $X \in T_mM$ the class $\{\widetilde{X} \in T_pP : p \in \pi^{-1}(m)\}$ of horizontal lifts, any one of which acts on vector-valued functions on P so as to define the value of the operator ∇_X at m (cf. (2.8)). In this way, sections of the vector bundle TP/H correspond to certain first order differential operators on M. Any two splittings differ by a section of Hom $(TM, \mathfrak{h}M)$ which measures the deviation $\widetilde{X}' - \widetilde{X}$ between the two horizontal lifts $\widetilde{X}' \in H'_p$, and $\widetilde{X} \in H_p$.

Given a connection form ϕ, and ξ as in (1.18), there is a unique connection form $\phi' = \phi + \theta\xi$ which inverts the above construction. The set of connections on an H-structure P is therefore an affine space modelled on $\Gamma(M, T^*M \otimes \mathfrak{h}M)$.

1.6 Proposition *Any H-structure P has a connection, and if P is integrable, it admits a torsion-free connection.*

Proof. A local section $s \in \Gamma(U, P)$ is tangent to a unique connection on $\pi^{-1}(U)$; if s is integrable, $s^* d\theta = 0$, and the connection is torsion-free. By above, the space of connections is certainly convex, so local connections can be patched together with a partition of unity, and torsion is additive. □

The existence of a torsion-free H-connection means that P is integrable "to first order". The Frobenius theorem provides a simple setting in which this is also a sufficient condition for integrability, H being the subgroup of $GL(n, \mathbb{R})$ preserving a p-dimensional subspace in $\mathbb{R}^n$. Other situations of varying difficulty for which the existence of a torsion-free connection implies integrability include almost symplectic and almost complex structures, discussed in chapter 3.

Once again, let P be an H-structure endowed with two distinct connections. From (1.12), $\Theta' - \Theta = [\phi' - \phi, \theta]$, and

$$T'(p)_{xy} - T(p)_{xy} = \tfrac{1}{2}(\xi(p)_x y - \xi(p)_y x), \quad x, y \in \mathbb{R}^n;$$

the first term signifies the result of applying the torsion to $x \wedge y$. In terms of the H-equivariant homomorphism

$$\delta\colon (\mathbb{R}^n)^* \otimes \mathfrak{h} \hookrightarrow (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \otimes \mathbb{R}^n \to \textstyle\bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n \tag{1.20}$$

built from the obvious inclusion and anti-symmetrization, we may write simply

$$T' - T = -\delta\xi.$$

Existence and uniqueness questions for H-connections on P are then settled by

1.7 Proposition *(i) If ϕ', ϕ are torsion-free, then $\xi(p) \in \ker\delta$;*
(ii) P has a torsion-free connection if and only if $T(p) \in \operatorname{Im}\delta$ for all $p \in P$.

The torsion $T(p)$ of any particular connection determines another equivariant function $T_0(p)$ on P with values in the quotient

$$\operatorname{coker}\delta \cong \frac{\bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n}{\delta((\mathbb{R}^n)^* \otimes \mathfrak{h})}. \tag{1.21}$$

By design, $T_0(p)$ is independent of the choice of connection, and is called the *structure function* of P; it is the obstruction to the existence of a torsion-free connection on P, or to first order integrability, and was introduced by Bernard [Ber]. In the language of Spencer cohomology, the space (1.21) is $H^{0,1}(\mathfrak{h})$; other cohomology groups house higher order obstructions to integrability, described by Guillemin [G], and Singer and Sternberg [SS].

Of particular interest is the case when $T_0(p)$ vanishes, or failing that when it is H-invariant, which means constant on the fibres of P. A conceptually important example is provided by the $\{e\}$-structure defined on a Lie group G, for which $P \cong M$, and $T_0(p) = T_0(m)$ is the Lie bracket $b \in \bigwedge^2 \mathfrak{g}^* \otimes \mathfrak{g}$.

2 Parallel Transport

The so-called fundamental theorem of Riemannian geometry asserts the existence and uniqueness of a metric connection whose torsion is zero. We explain how this works using the terminology of the preceding pages, and point out that its proof gives valuable information when the manifold has a structure defined by a *subgroup* of $O(n)$. In practice, such a subgroup is often furnished by the holonomy group, generated by parallel transport along closed curves. This transport is most easily described in terms of the horizontal subspaces that arise as the kernel of the connection form at each point of the principal bundle of orthonormal frames.

In a certain sense, the holonomy group is also generated by the Riemann curvature tensor R, which is a measure of infinitesimal parallel transport, but the relationship between curvature and holonomy is quite subtle. In the course of this chapter, we shall begin to familiarize ourselves with R, but a more systematic account of its properties will appear in due course. These properties impose severe restrictions on the holonomy group of a Riemannian manifold, restrictions which do not apply for arbitrary connections, even ones with zero torsion [HO].

Part of the theory is valid for a pseudo-Riemannnian manifold with an indefinite metric. However one feature special to the positive definite case is that the metric is locally a product if and only if H acts reducibly on the tangent space. This fact implies, amongst other things, that the identity component of H is compact. Other essential properties of holonomy groups are discussed briefly, but for detailed proofs of the major theorems, and more exhaustive definitions, we refer the reader to Kobayashi and Nomizu's treatment [KN, chapter 2 and appendices 4,5,7].

The Levi Civita connection

We shall now suppose that M is an n-dimensional Riemannian manifold with metric tensor g, as described in (1.6). The corresponding principal bundle P of orthonormal frames constitutes an $O(n)$-structure. The structure group $O(n)$ preserves an inner product on the vector space $\mathbb{R}^n$, which induces an isomorphism $\mathbb{R}^n \cong (\mathbb{R}^n)^*$. In classical notation, this corresponds to the *index lowering* operation $a^i \mapsto g_{ij}a^j$, where a^i denote the coefficients of a vector relative to a basis of $\mathbb{R}^n$, and $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$

are the coefficients of the metric tensor. Moreover

$$\begin{aligned} \operatorname{End}\mathbb{R}^n = (\mathbb{R}^n)^*\otimes\mathbb{R}^n &\cong \Lambda^1\otimes\Lambda^1 \\ &= \textstyle\bigodot^2(\Lambda^1)\oplus\bigwedge^2(\Lambda^1) \\ &= \mathbb{R}\oplus\Sigma_0^2\oplus\Lambda^2. \end{aligned} \tag{2.1}$$

In general, we shall use Σ^k, Λ^k as abbreviations for the spaces $\bigodot^k(\mathbb{R}^n)^*$, $\bigwedge^k(\mathbb{R}^n)^*$ of totally symmetric and skew-symmetric tensors respectively. In (2.1), the identity endomorphism corresponds to the inner product, and the space of traceless self-adjoint endomorphisms is identified with the space Σ_0^2 of symmetric 2-tensors orthogonal to the inner product. More importantly, there is an isomorphism

$$\mathfrak{so}(n) \cong \Lambda^2, \tag{2.2}$$

between the Lie algebra of skew-adjoint endomorphisms and the space of 2-forms.

Given the Lie algebra $\mathfrak{h} = \mathfrak{so}(n)$, the fundamental homomorphism (1.20) that sends the difference of two connections to the difference of their torsions is the natural mapping

$$\delta\colon \Lambda^1\otimes\Lambda^2 \longrightarrow \Lambda^2\otimes\Lambda^1. \tag{2.3}$$

Using indices, an element a_{ijk} of $\Lambda^1\otimes\Lambda^2$ can be thought of as the difference of the respective Christoffel symbols of two connections, and $\delta(a_{ijk}) = \frac{1}{2}(a_{ijk} - a_{jik})$. The amusing formula

$$a_{ijk} = a_{jik} = -a_{jki} = -a_{kji} = a_{kij} = a_{ikj} = -a_{ijk},$$

valid for $a_{ijk} \in \ker\delta$, confirms that δ is an isomorphism. Consequently,

2.1 Proposition *An $O(n)$-structure has a unique torsion-free connection.*

This connection is called the *Levi Civita* or *Riemannian* connection. We shall refer to it with either of the two symbols ϕ, ∇, which stand respectively for its connection form on P and its covariant differentation.

The above theory can be carried out in almost the same way for the pseudo-orthogonal group $O(p,q)$, using the analogue $\mathfrak{so}(p,q) \cong \Lambda^2$ of (2.2), and there is a unique torsion-free connection preserving the indefinite metric. In fact, suppose that H is a closed subgroup of $GL(n,\mathbb{R})$, $n \geqslant 3$, for which *any* H-structure on *any* manifold admits a unique torsion-free connection. Then a theorem of Weyl, described in [C_2], asserts that necessarily $\mathfrak{g} \cong \mathfrak{so}(p,q)$. Other groups (like the quaternionic linear group $GL(k,\mathbb{H})$ which will be mentioned again in chapter 8) give rise to structures in which existence of a torsion-free connection is not guaranteed, but if it does exist it is unique.

There are various equivalent ways to express the fact that the Levi Civita connection reduces to the $O(n)$-bundle P. From **1.3**, the metric tensor g is parallel, which means that

$$
\begin{aligned}
\frac{\partial g_{ij}}{\partial x^k} &= \frac{\partial}{\partial x^k}\Big(g(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})\Big)\\
&= g(\nabla_{\partial/\partial x^k}\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}) + g(\frac{\partial}{\partial x^i},\nabla_{\partial/\partial x^k}\frac{\partial}{\partial x^j})\\
&= \sum_r\Big(\Gamma^r_{ki}g_{rj} + g_{ir}\Gamma^r_{kj}\Big).
\end{aligned}
\tag{2.4}
$$

The symmetry of the Christoffel symbols (recall (1.16)) then leads to the explicit formula

$$\sum_r \Gamma^r_{ij}g_{rl} = \tfrac12\Big(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\Big).$$

Given any orthonormal frame $p\in P$, let D_p denote the horizontal subspace of T_pP determined by the Levi Civita connection. By **1.1**, there exists an integrable section $s = \{\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}\}$ of LM *tangent* to D_p at p. Not only are the elements of s orthonormal at $m=\pi(p)$, but they are covariant constant at m; thus

$$
\begin{aligned}
&g_{ij}\big|_m = \delta_{ij},\\
&\frac{\partial g_{ij}}{\partial x^k}\bigg|_m = \tfrac12(\Gamma^i_{kj}|_m + \Gamma^j_{ki}|_m) = 0.
\end{aligned}
\tag{2.5}
$$

Of course, the coordinates $x^1,\dots,x^n$ need not necessarily be *normal* ones that are defined by means of geodesics.

Now suppose that H is a closed subgroup of $O(n)$, equal to the stabilizer of some element η in an $O(n)$-module V, like the situation underlying **1.3**. Typically V will be an exterior power Λ^k of the basic representation, and η will give rise to a differential form on the manifold. Let $\mathfrak{h}$ denote the Lie algebra of H, and $\mathfrak{h}^\perp$ its orthogonal complement in $\mathfrak{so}(n)\cong\Lambda^2$. Restricting to the subalgebra $\mathfrak{h}$ gives an updated version of (1.20), namely a monomorphism

$$\delta\colon \Lambda^1\otimes\mathfrak{h} \longrightarrow \Lambda^2\otimes\Lambda^1,$$

for which

$$\text{coker}\delta \cong (\text{Im}\delta)^\perp \cong \Lambda^1 \otimes \mathfrak{h}^\perp. \tag{2.6}$$

Recall that the H-structure function $T_0(p)$ is defined to be the component in (2.6) of the torsion of any H-connection ϕ'. In view of **2.1**, this torsion is effectively the tensor ξ with values in $\Lambda^1 \otimes \mathfrak{so}(n)$, for which $\theta\xi = \phi' - \phi$. The differential of the action of $O(n)$ on η defines a linear map $\mathfrak{so}(n) \to V$ with kernel $\mathfrak{h}$. Thus $\mathfrak{h}^\perp$ is embedded in V, and $(\nabla\eta)(p) = -\xi(p)(\eta)$ belongs to $\Lambda^1 \otimes \mathfrak{h}^\perp$ for each $p \in P$. In summary,

2.2 Corollary *The obstruction $T_0(p)$ to the existence of a torsion-free H-connection can be identified with $(\nabla\eta)(p)$, and has values in the space $\Lambda^1 \otimes \mathfrak{h}^\perp$.*

The value of **2.2** is enhanced by the knowledge that in many useful cases, the representation $\mathfrak{h}^\perp$ turns out to be *irreducible*; for example, this is so when H is simple and the isotropy subgroup of an irreducible symmetric space (see also (5.17)). Notice that for any closed subgroup $H \subset O(n)$, there exists a unique H-connection with torsion $T(p) = T_0(p)$; it is sometimes convenient to work with this "normalized" connection instead of the Riemannian one. The structure function $T_0(p)$ occurring in **2.2** is described in similar terms by Bryant [Br_2] as the *intrinsic torsion* of the H-structure.

Horizontality

Curvature measures non-integrability of the horizontal distribution on the bundle P of orthonormal frames. Indeed, if P' is any submanifold of P whose tangent spaces are contained in the horizontal distribution, then the pullback of the 2-form Φ to P' is zero (see (1.13)). Of course, there is no obstruction to the existence of horizontal 1-dimensional submanifolds, or *curves.*

2.3 Lemma *If $\gamma\colon (a,b) \to M$ is a smooth curve, and $p \in \pi^{-1}(\gamma(a))$, then there exists a unique smooth lift $\tilde\gamma\colon (a,b) \to P$, $\pi\circ\tilde\gamma = \gamma$, with $\tilde\gamma(a) = p$.*

Proof. It suffices to consider the case in which γ is embedded in M; in this case the result follows by restricting attention to the manifold $\pi^{-1}(\gamma)$ equipped with its integrable horizontal fields. Put more rigorously, the induced bundle $\gamma^{-1}P$ over

$[a,b]$ has a section $\widetilde{\gamma} = \{X_1, \ldots, X_n\}$ made up of vector fields X_r along γ such that $\nabla_Z X_r = 0$, where $Z = \dot{\gamma}(t)$ is tangent to γ, and ∇ is the induced connection whose curvature is zero. In terms of local coordinates, $Z = \sum_j \frac{d\gamma^j}{dt}\frac{\partial}{\partial x^j}$, and

$$\frac{dX_r^i}{dt} + \sum_{j,k} \Gamma_{jk}^i \frac{d\gamma^j}{dt} X_r^k = 0, \quad 1 \leqslant i, r \leqslant n, \tag{2.7}$$

has unique solutions $X_r = \sum_i X_r^i \frac{\partial}{\partial x^i}$, given the initial value p. □

Because the horizontal distribution is equivariant, if $\tilde{\gamma}$ is a horizontal curve, then so is its right translate $R_g \circ \tilde{\gamma}$. It follows that for every $t \in [a,b]$, the operation

$$\Pi_t \;=\; \tilde{\gamma}(t) \circ (\tilde{\gamma}(a))^{-1} \colon T_{\gamma(a)}M \to T_{\gamma(t)}M$$

of *parallel transport* is an isometry that depends only on the projection $\gamma = \pi \circ \widetilde{\gamma}$. Given a tangent vector $Z \in T_m M$, choose a curve γ with $\gamma(a) = m$ and $\dot{\gamma}(a) = Z$. Then for any vector field X on M,

$$\nabla_Z X \;=\; \left.\frac{d}{dt}\right|_{t=a} \Pi_t^{-1} X. \tag{2.8}$$

This is just a restatement of the definition of (1.10) of the covariant derivative as differentiation in the horizontal directions of the principal frame bundle.

Define two points p, q of P to be *equivalent* $(p \sim q)$ if there exists a horizontal curve $\tilde{\gamma} \colon (a,b) \to P$ with $\tilde{\gamma}(a) = p$ and $\tilde{\gamma}(b) = q$. From now on, we allow *piece-wise* smooth curves, in order that $\sim$ is obviously an equivalence relation. Fix an orthonormal frame $p \in \pi^{-1}(m) \subset P$, and let

$$Q(p) = \{q \in P : p \sim q\}$$

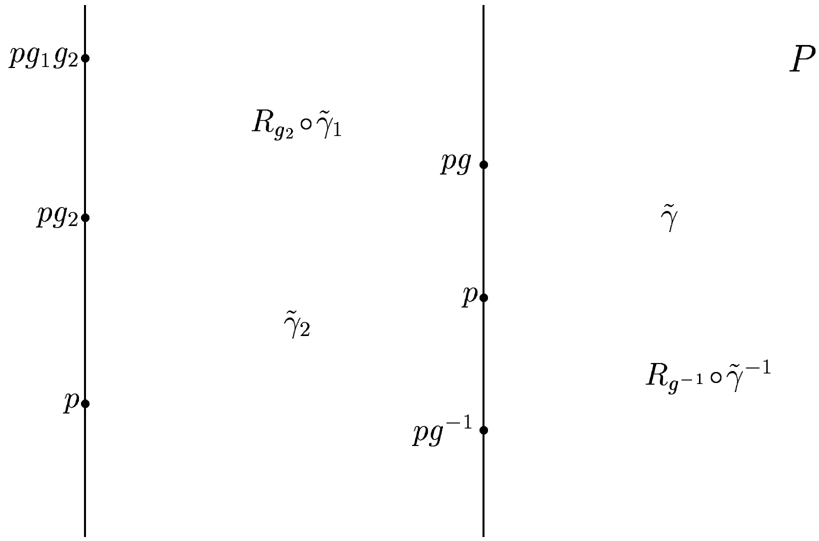
denote the equivalence class consisting of all frames obtained from p by parallel transport.

Of special interest is the set of frames

$$H(p) = \{g \in O(n) : pg \in Q(p)\} = Q(p) \cap \pi^{-1}(m)$$

in the same fibre as p. Thus g belongs to $H(p)$ if and only if there is a loop $\gamma \colon (a,b) \to M$ with $\gamma(a) = m = \gamma(b)$, $\tilde{\gamma}(a) = p$ and $\tilde{\gamma}(b) = q$. The usual operations on loops ensure that $H(p)$ is a subgroup of $O(n)$; it is called the *linear holonomy group with reference point* p.

2.4 Figure Product and inverse in $H(p)$



If $p \sim q$, then clearly $H(p) = H(q)$; furthermore,

$$H(pg) = g^{-1}H(p)g, \quad g \in O(n).$$

When the reference point is not important, we shall denote the holonomy group simply by H; as such it is well defined up to conjugation in $O(n)$. The interest expressed in chapter 1 in H-structures and H-connections stems from the following "Reduction Theorem".

2.5 Proposition *$Q(p)$ is an $H(p)$-structure to which the Levi Civita connection reduces.*

Proof. Each fibre of $Q(p)$ is isomorphic to $H(p)$, and one has to exhibit smooth *local sections* of the equivalence class $Q(p)$. Such a section passing through p is generated by the horizontal lifts of radial curves with respect to a coordinate system centred at $m = \pi(p)$. The tangent to this section at p will equal the horizontal subspace D_p defined by the Levi Civita connection. The latter therefore reduces to $Q(p)$. $\square$

The principal subbundle $Q(p)$ is called the *holonomy bundle* with reference to the fixed frame p; changing p obviously leads to an isomorphic bundle. In fact, $Q(p)$ is a maximal integral submanifold of the distribution $\{T_q(Q(q)) : q \in LM\}$, and this implies that the holonomy groups defined by working with curves of class C^k,

$1 \leqslant k \leqslant \infty$, all coincide. We refer the reader to [KN, chapter 2, sections 6,7] or [NO] for more details.

There are in fact distinguished sections of $Q(p)$, constructed canonically from the so-called *basic vector fields* on LM. An element $x \in \mathbb{R}^n$ gives rise naturally to a vector field X on LM, whose value at a frame q is defined to be the unique horizontal vector $X|_q$ for which $qx = \pi_*(X|_q)$. The projection $\gamma = \pi \circ \widetilde{\gamma}$ of any integral curve $\widetilde{\gamma}$ of X has the property that its tangent vector is parallel along itself. Such a curve γ is a *geodesic* on M; from (2.7) it is a solution of the equation

$$\frac{d^2\gamma^i}{dt^2} + \sum_{j,k} \Gamma^i_{jk} \frac{d\gamma^j}{dt}\frac{d\gamma^k}{dt} = 0.$$

The Riemannian manifold M is *complete* if the length parameter t of every geodesic extends to $(-\infty, \infty)$; in this case any two points may be joined by at least one geodesic [HR].

Fix a point m in M. Provided the vector x is sufficiently small, it will determine another point in M which lies a unit parameter distance along γ from m. As x varies, this process sets up the *exponential mapping* from a neighbourhood of 0 in $\mathbb{R}^n$ to a neighbourhood of m. The components $x^1, \ldots, x^n$ of the vector x then constitute the distinguished system of *normal coordinates* centred at m, and satisfying (2.5). These coordinates give rise to two distinct sections of LM, which are both tangent to D_p at p. One is the integrable section $s = \{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$, and the other is the local section of $Q(p)$, swept out by the basic integral curves $\widetilde{\gamma}$ passing through p.

For the remainder of this chapter, we fix an orthonormal frame p. Let $\pi_1(M)$ denote the fundamental group of M, consisting of equivalence classes of homotopic loops based at $\pi(p)$. Any element of $\pi_1(M)$ may be represented by a piecewise smooth loop γ, which determines an element in $H = H(p)$. If $H^0 = H^0(p)$ denotes the normal subgroup of H formed by parallel transport along null-homotopic loops, we obtain an epimorphism

$$\pi_1(M) \longrightarrow {}^{H}\!/_{H^0}. \tag{2.9}$$

The subgroup H^0 is itself called the *restricted holonomy group*. Clearly H^0 is a subgroup of $SO(n)$, and (2.9) implies that H/H^0 is countable.

The advantage of a null-homotopic loop is that, as shown by Lichnerowicz [L_1], it may be factorized into a product of "lassos" $\gamma_1^{-1} \circ \gamma_0 \circ \gamma_1$, where γ_0 lies in the domain of a coordinate chart and γ_1 forms half of the "noose" originating from $\pi(p)$. One can then deduce that any element of H^0 can be joined to the identity by a family of *piecewise smooth* curves that lie in H^0. Using a theorem of Kuranishi and Yamabe [Ya], one obtains the following corollary, which we shall take for granted:

2.6 Theorem *H is a Lie subgroup of $O(n)$, whose identity component is H^0.*

The Riemann curvature tensor

If the curvature tensor $R(p)$ of the Levi Civita connection vanishes, then the tensors Φ, Θ are both identically zero on the frame bundle LM, and $\phi+\theta$ satisfies the Maurer-Cartan equations (1.4) for the affine group. Another way of saying this is provided by the well-known

2.7 Proposition *The curvature tensor of a Riemannian manifold M vanishes identically if and only if its $O(n)$-structure P is integrable.*

Proof. If $\Phi=0$, the distribution of horizontal spaces D_p on P is integrable, and any point of M has a neighbourhood U with a horizontal section $s\in\Gamma(U,P)$. Then $s^*d\theta=s^*\Theta=0$, and on a possibly smaller neighbourhood U', the section s arises as a coordinate-induced orthonormal basis $\{\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}\}$. □

The above proposition may be rephrased by saying that the restricted holonomy group H^0 of M reduces to the identity if and only if the metric can be put locally in standard form $\sum_{i=1}^n dx^i\otimes dx^i$; in either case M is said to be *flat*. Let M be a complete, connected flat Riemannian manifold. Parallel transport then trivializes the frame bundle LM over any simply-connected domain, and the universal covering space of M is isomorphic to $\mathbb{R}^n$. It follows that M is itself the quotient of $\mathbb{R}^n$ by a discrete group of isometries, which can be identified with $\pi_1(M)$. Moreover, it is not hard to see that the kernel of the *holonomy presentation*

$$\pi_1(M)\longrightarrow H,$$

given by (2.9), equals the subgroup N consisting of pure translations.

In the above situation, the fundamental group $\pi_1(M)$ acts properly discontinously on $\mathbb{R}^n$. If M is also compact, $\pi_1(M)$ is a so-called crystallographic group, and by Bieberbach's theorem [Bi], the holonomy group $H\cong\pi_1(M)/N$ is finite. Thus there is a finite covering

$$\mathbb{R}^n/N\longrightarrow\mathbb{R}^n/\pi_1(M)=M, \tag{2.10}$$

of M by a flat torus, which itself has trivial holonomy group. For more details we refer the reader to [KN, chapter 5, theorem 4.2], or [W_3]. Auslander and Kuranishi [AK] have shown that any finite group H is known to arise as the holonomy group of a suitable flat compact Riemannian manifold.

Let q be an orthonormal frame in the holonomy bundle $Q = Q(p)$, so that it is joined to the fixed frame p by a horizontal curve. Because the connection reduces to $Q(p)$, the curvature operator $R(q)_{xy} = R(q)(x \wedge y)$ defined by (1.13) belongs to the holonomy algebra $\mathfrak{h}$, for any $x, y \in \mathbb{R}^n$. This fact may be understood intuitively by choosing vector fields X, Y on a neighbourhood of $m = \pi(q)$, with $X|_m = qx$, $Y|_m = qy$, and $(\nabla X)|_m = 0 = (\nabla Y)|_m$. Then **1.2** implies that $R(q)_{xy}$ equals the value of the commutator

$$[\nabla_X, \nabla_Y] \;=\; \nabla_X \nabla_Y - \nabla_Y \nabla_X$$

at q. The formula (2.8) then interprets $R(q)_{xy}$ as an infinitesimal measure of the non-commutativity of parallel transport in the directions determined by X and Y.

The links between curvature and holonomy are made more precise by the following "Holonomy Theorem", due to Cartan [C_3] and Ambrose and Singer [AmS], which asserts that the Lie algebra $\mathfrak{h}$ is actually determined by the values of the curvature function $R(q)$ as q ranges over Q. To be more precise, we regard $R(q)$ as a map from Λ^2 to $\mathfrak{so}(n)$, and consider its image $\mathrm{Im}(R(q)) = \{R(q)_{xy} : x, y \in \mathbb{R}^n\}$. In these terms, we have

2.8 Theorem $\mathfrak{h} = \mathrm{span}\{\mathrm{Im}(R(q)) : q \in Q\}$.

Proof. First note that

$$\begin{aligned} \mathfrak{h}_m &= \mathrm{span}\{\mathrm{Im}(R(q)) : q \in \pi^{-1}(m) \cap Q\} \\ &= \mathrm{span}\{(\mathrm{Ad}\, h)\mathrm{Im}(R(q)) : h \in H(q)\}, \end{aligned} \qquad (2.11)$$

is an ideal of $\mathfrak{h}$, where (just) for the last line q is fixed.

The standard proof then goes on to consider the span $\mathfrak{h}'$ of the $\mathfrak{h}_m$, $m \in M$, which is conceivably a proper ideal of $\mathfrak{h}$. The key point is that the 2-form

$$\Phi = d\phi + \tfrac{1}{2}[\phi, \phi]$$

on Q takes values in $\mathfrak{h}'$, and it follows immediately from the Frobenius theorem that the distribution $\{X \in TQ : \phi(X) \in \mathfrak{h}'\}$ is integrable. Since this distribution engulfs the horizontal subspaces D_p which are annihilated by ϕ, a maximal integral submanifold Q' contains any horizontal curve emanating from p. Therefore $Q' = Q$, which forces $\mathfrak{h}' = \mathfrak{h}$. □

If one knew, for example, that $\mathfrak{h}$ were a *simple* Lie algebra, then $\mathfrak{h}_m$ of $\mathfrak{h}$ would either be zero, or equal to $\mathfrak{h}$. One is tempted to call the subalgebra $\mathfrak{h}_m$ the "pointwise holonomy algebra" at m, but its definition still requires knowledge of the holonomy group itself. Another approximation to $\mathfrak{h}$ is given by the span of all covariant derivatives of the curvature tensor, evaluated at a fixed $q \in Q$. This defines the so-called *infinitesimal holonomy algebra* $\mathfrak{h}^{\mathrm{inf}}(q)$, and sandwiched between $\mathfrak{h}^{\mathrm{inf}}(q)$ and $\mathfrak{h}$ is the *local holonomy algebra* $\mathfrak{h}^{\mathrm{loc}}(q)$ defined in the same way as $\mathfrak{h}$, but using arbitrarily small neighbourhoods of m. These notions were introduced by Nijenhuis [N], who proved that in real analytic case, $\mathfrak{h}^{\mathrm{inf}} = \mathfrak{h}^{\mathrm{loc}} = \mathfrak{h}$.

Decomposable Metrics

The action of the holonomy group $H = H(p)$ provides an invaluable guide to the behaviour of the Riemannian metric. A striking illustration of this occurs when H acts reducibly on the tangent space representation, so that $\mathbb{R}^n$ contains a proper subspace V invariant by H. Such a subspace V gives rise, in the presence of a positive definite metric, to complementary invariant subspace $V^\perp$, so at the end of the day there is always a direct sum

$$\mathbb{R}^n = V_0 \oplus V_1 \oplus \cdots \oplus V_k, \tag{2.12}$$

in which H acts *trivially* (i.e., as the identity) on V_0, which may or may not be zero, and H acts *irreducibly* on each V_k for $k \geqslant 1$.

2.9 Proposition *Under the assumption* (2.12), *the restricted holonomy group* H^0 *is isomorphic to a product*

$$\{e\} \times H_1 \times \cdots \times H_k, \tag{2.13}$$

and M *is locally isomorphic to a Riemannian product*

$$M_0 \times M_1 \times \cdots \times M_k,$$

with M_0 *flat.*

Proof. Relative to a frame $q \in Q(p)$, the curvature operator $R_{xy} = R(q)_{xy}$ has values in $\mathfrak{h}$, and so

$$\begin{aligned} R_{xy}\Big|_{V_0} &= 0, \\ R_{xy}(V_i) &\subseteq V_i, \quad 1 \leqslant i \leqslant k. \end{aligned} \tag{2.14}$$

Let $x = \sum x_i$, $y = \sum y_i$, so that x_i and y_i are the components of x and y in V_i. In view of (2.14), it would be convenient to suppose that

$$R_{xy} = \sum_{i=1}^{k} R_{x_i y_i};$$

in fact this equation is an immediate consequence of the well-known symmetry

$$g(R_{vw}x, y) = g(R_{xy}v, w)$$

that will be discussed fully in chapter 4.

As q ranges over $\pi^{-1}(m) \cap Q(p)$, and x, y over $\mathbb{R}^n$, the operators $R_{x_i y_i}$ span an ideal $(\mathfrak{h}_i)_m \subset \operatorname{End} V_i$ of $\mathfrak{h}$. Varying m then leads to a decomposition

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_k,$$

whence (2.13).

The proof of the second part of the proposition, that we shall sketch, is based on a geometrical interpretation of the preceding algebra. Because V_i is H-invariant, it gives rise via parallel translation to a well-defined distribution $V_iM \subset TM$, with the property that $\nabla_X Y$ belongs to V_iM whenever X, Y do. It follows from (1.17) and the Frobenius theorem that V_iM is integrable. The condition (2.14) is then the infinitesimal counterpart of the fact that any integral manifold of V_iM is *totally geodesic*, which means that geodesics initially tangent to it remain so.

On a neighbourhood of m, we may now choose coordinates

$$x_1^1, \ldots, x_1^{r_1}; \ \ldots \ ; x_k^1, \ldots, x_k^{r_k}$$

with $\frac{\partial}{\partial x_i^1}, \ldots, \frac{\partial}{\partial x_i^{r_i}}$ all tangent to V_iM. If $X = \frac{\partial}{\partial x_i^r}$, $Y = \frac{\partial}{\partial x_j^s}$ with $i \neq j$, then $\nabla_X Y = \nabla_Y X$ belongs to $V_iM \cap V_jM = \{0\}$. Using (2.4), one deduces that the component of the metric tensor corresponding to V_iM is annihilated by $\frac{\partial}{\partial x_j^s}$ whenever $j \neq i$, and is a function only of $x_i^1, \ldots x_i^{r_i}$. It then suffices to take M_i to be the maximal integral submanifold of V_iM through m. □

The decomposition theorem of de Rham theorem [dR] is a global version of **2.9** which asserts that a complete, simply-connected Riemannian manifold whose holonomy group $H = H^0$ acts reducibly is a Riemannian product, whose factors consist of the maximal integral manifolds M_i. In this case, it is clear that each subgroup H_i that arises is the holonomy group of the Riemannian manifold M_i. In general, it makes sense to use the terminology *locally irreducible* for a Riemannian manifold whose holonomy representation on $\mathbb{R}^n$ is irreducible.

Now a connected Lie subgroup of $SO(n)$ acting irreducibly on $\mathbb{R}^n$ is known to be a closed subgroup. This fact, when combined with the "complete splitting" of **2.9**, has the following important consequence.

2.10 Theorem [BL_1] *The restricted holonomy group H^0 of an n-dimensional Riemannian manifold is compact.*

An open problem is to determine under what circumstances the (unrestricted) holonomy group H of a *compact* Riemannian manifold M is itself compact. The definitions imply that H is contained in the normalizer of H^0, so the latter must not be "too big" if H is to be non-compact. At the other extreme, we have already explained that if $H^0 = \{e\}$, then H is finite. In fact, if M is compact and connected with $\dim V_0 \leqslant 1$ in (2.12), then H is compact (see, for example, [Bes; 10.117]).

3 The Unitary Holonomy Group

The most familiar non-generic holonomy group is the maximal subgroup $U(m)$ of $SO(2m)$. Metrics whose holonomy group is contained in $U(m)$ are called Kähler; their study blends together complex and Riemannian geometry, and they arise naturally from both local and global considerations. For example, if the derivatives $\partial^2 f/\partial z^\alpha \partial \overline{z}^\beta$ of a real function f on an open set of $\mathbb{C}^m$ define a positive definite matrix, they constitute the coefficients of a Kähler metric. In algebraic geometry Kähler metrics arise on submanifolds of the complex projective space, which is itself the fundamental compact example.

Throughout this chapter we shall be dealing with a manifold of real dimension $n = 2m$. The crucial algebraic concept is that of an almost complex structure, which in the present context originates from the centre of the Lie algebra $\mathfrak{u}(m)$ of skew-Hermitian matrices, and gives rise also to a non-degenerate 2-form. It is then appropriate to consider both complex and symplectic manifolds, both of which are introduced using the formalism of torsion and structure functions. After some simple constructions of Kähler manifolds, the chapter concludes with remarks and generalities concerning the Dolbeault complex.

Hermitian algebra

The unitary group $U(m)$ may be defined as the set of complex linear transformations of $\mathbb{C}^m$ preserving the Hermitian form $\eta = \sum_{\alpha=1}^m dz^\alpha \otimes \overline{dz}^\alpha$. We may regard the forms dz^α as elements of the dual space $(\mathbb{C}^m)^*$, which we denote by $\lambda^{1,0}$, and $\overline{dz}^\alpha$ as elements of its conjugate $\lambda^{0,1} = (\overline{\mathbb{C}^m})^*$. More generally, we set

$$\lambda^{p,q} = \textstyle\bigwedge^p (\mathbb{C}^m)^* \otimes \bigwedge^q (\overline{\mathbb{C}^m})^*,$$

so that η is an invariant element of the space $\lambda^{1,1}$. Actually, it is not necessary to be quite so fussy with the notation, since η defines an isomorphism $\overline{\mathbb{C}^m} \cong (\mathbb{C}^m)^*$, and a bar and an asterisk cancel each other out.

Of primary interest is the action of $U(m)$ on the *real* vector space Λ^1 underlying both $\lambda^{1,0}$ and $\lambda^{0,1}$. In real coordinates, the Hermitian form is

$$\begin{aligned}\eta &= \sum_{\alpha=1}^{m}(dx^\alpha + idy^\alpha)\otimes(dx^\alpha - idy^\alpha)\\ &= \sum(dx^\alpha\otimes dx^\alpha + dy^\alpha\otimes dy^\alpha) - 2i\sum dx^\alpha\wedge dy^\alpha.\end{aligned} \tag{3.1}$$

The real part $g = \operatorname{Re}\eta$ is the standard inner product on $\mathbb{R}^{2m}$, so that $U(m)$ acts on Λ^1 as a subgroup of $O(2m)$. The unitary group $U(m)$ also commutes with the real endomorphism I of $\mathbb{R}^{2m}$ satisfying $Idx^\alpha = -dy^\alpha$, $Idy^\alpha = dx^\alpha$, which is induced from multiplication by i on $\lambda^{1,0}$, or equivalently $-i$ on $\lambda^{0,1}$. If we order the coordinates of $\mathbb{R}^{2m}$ as $(x^1,\dots,x^m,y^1,\dots,y^m)$, then the inclusion $U(m)\hookrightarrow O(2m)$ is given in terms of matrices by

$$X = A + iB \mapsto \begin{pmatrix} A & -B\\ B & A\end{pmatrix}, \tag{3.2}$$

and $X\overline{X}^t = \mathbf{1}$ translates into $AA^t + BB^t = \mathbf{1}$ and $AB^t - BA^t = \mathbf{0}$. Since $U(m)$ is connected, its image must actually lie in $SO(2m)$, and the determinant of the right-hand side of (3.2) equals one. The main point is that the image of the scalar matrix $i\mathbf{1}$ may be identified with $I \in SO(2m)\cap\mathfrak{so}(2m)$.

Any endomorphism I of a real vector space with I^2 equal to minus the identity is called an *almost complex structure*. Notice that I plays a different role to the symbol i in (3.1), which identifies the complexification

$$\Lambda^1\oplus i\Lambda^1 = \Lambda^1\otimes_{\mathbb{R}}\mathbb{C} = \lambda^{1,0}\oplus\lambda^{0,1}.$$

of Λ^1. Taking exterior powers of the last equation gives the decomposition of complexified k-forms according to type:

$$\Lambda^k\otimes_{\mathbb{R}}\mathbb{C} \cong \bigoplus_{p+q=k}\lambda^{p,q}, \quad 0\leqslant k\leqslant 2m. \tag{3.3}$$

As a space of forms, $\lambda^{p,q}$ is spanned by $dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge\overline{dz}^{j_1}\wedge\cdots\wedge\overline{dz}^{j_q}$. However, because of our ultimate interest in real quantities, it is convenient to define *real* vector spaces $[\![\lambda^{p,q}]\!]$, $[\lambda^{p,p}]$ by

$$\begin{aligned}&[\![\lambda^{p,q}]\!]\otimes_{\mathbb{R}}\mathbb{C} = \lambda^{p,q}\oplus\lambda^{q,p}, \quad p\neq q,\\ &[\lambda^{p,p}]\otimes_{\mathbb{R}}\mathbb{C} = \lambda^{p,p}.\end{aligned} \tag{3.4}$$

If $\mathfrak{u}(m)^\perp$ denotes the orthogonal complement of $\mathfrak{u}(m)$ as a subalgebra of $\mathfrak{so}(2m)$, then in view of (2.2), the following decompositions are essentially equivalent:

$$\begin{aligned} \Lambda^2 &= [\lambda^{1,1}] \oplus [\![\lambda^{0,2}]\!] \\ \mathfrak{so}(2m) &= \mathfrak{u}(m) \oplus \mathfrak{u}(m)^\perp. \end{aligned} \tag{3.5}$$

As a real subspace of 2-forms, $[\lambda^{1,1}]$ contains the invariant element

$$\omega = \operatorname{Im}\eta = -i \sum_\alpha dz^\alpha \wedge \overline{dz}^\alpha, \tag{3.6}$$

which is $-i$ times the anti-symmetrization of η. The orthogonal complement $[\lambda_0^{1,1}]$ of ω in $[\lambda^{1,1}]$ can be identified with the Lie algebra $\mathfrak{su}(m)$. As a bilinear form, ω is *non-degenerate*, which is equivalent to the assertion that $\omega^m \neq 0$.

Wedging with ω determines an $U(m)$-equivariant mapping $L\colon \lambda^{p-1,q-1} \to \lambda^{p,q}$, and there is a well-known theory that develops properties of L and its adjoint, by regarding them as generators of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ [We]. All we shall need to know is that L is injective, provided $p+q \leqslant m$. For this range of p,q, the space $\lambda_0^{p,q}$ of so-called *primitive forms* can then be defined by the Hermitian direct sum $\lambda^{p,q} = \lambda_0^{p,q} \oplus L(\lambda^{p-1,q-1})$, so that $\lambda^{p,0} = \lambda_0^{p,0}$, and

3.1 Lemma *For $p \geqslant q$ and $p+q \leqslant m$, there is a $U(m)$-decomposition*

$$\lambda^{p,q} \;\cong\; \lambda_0^{p,q} \oplus \lambda_0^{p-1,q-1} \oplus \cdots \oplus \lambda_0^{p-q+1,1} \oplus \lambda^{p-q,0}.$$

In practice, this provides the following expressions for the exterior algebra of the tangent space in terms of irreducible $U(m)$ spaces, all of which are non-zero only when $m \geqslant 5$:

$$\begin{aligned} \Lambda^0 &\cong [\lambda^{0,0}], \\ \Lambda^1 &\cong [\![\lambda^{1,0}]\!], \\ \Lambda^2 &\cong [\![\lambda^{2,0}]\!] \oplus [\lambda_0^{1,1}] \oplus \Lambda^0, \\ \Lambda^3 &\cong [\![\lambda^{3,0}]\!] \oplus [\![\lambda_0^{2,1}]\!] \oplus \Lambda^1, \\ \Lambda^4 &\cong [\![\lambda^{4,0}]\!] \oplus [\![\lambda_0^{3,1}]\!] \oplus [\lambda_0^{2,2}] \oplus \Lambda^2, \\ \Lambda^5 &\cong [\![\lambda^{5,0}]\!] \oplus [\![\lambda_0^{4,1}]\!] \oplus [\![\lambda_0^{3,2}]\!] \oplus \Lambda^3, \end{aligned} \tag{3.7}$$

etc.

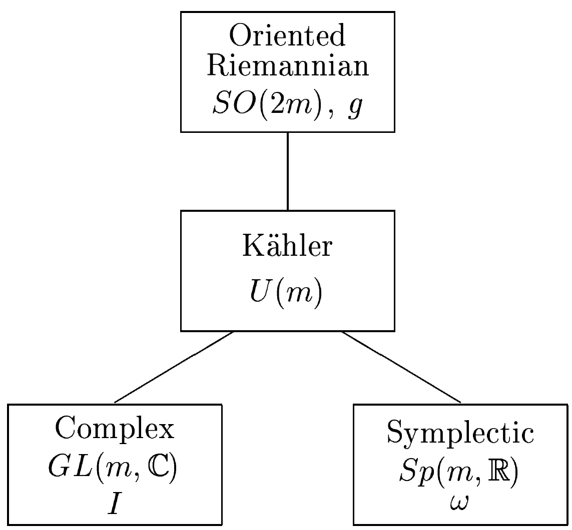
Complex and Kähler manifolds

Suppose that M is a manifold of even real dimension $2m$ with a $U(m)$-structure $P \subset LM$. Any frame $p \in P$ gives isomorphisms $\mathbb{R}^{2m} \cong T_m M$, $\Lambda^1 \cong T_m^* M$, and permits all the preceding algebra to be transferred to the tangent and cotangent spaces of M. In particular g, I and ω extend to equivariant functions on LM, and define tensors on M, satisfying the identity

$$g(X,Y) = \omega(IX,Y) = g(IX,IY). \tag{3.8}$$

The transformation I is both orthogonal and self-adjoint relative to the metric g, and determines the homotopy class of the reduction to $U(m)$.

3.2 Figure Structures associated to subgroups of $GL(2m,\mathbb{R})$



If η denotes one of the three tensors g, I, ω, its stabilizer H in $GL(2m,\mathbb{R})$ is the structure group of a unique subbundle of LM containing P. Moreover, $U(m)$ and P are recovered as the intersection of any two of these three groups or H-structures. The stabilizer of the almost complex structure I is the group $GL(m,\mathbb{C})$ of complex linear transformations, embedded in $GL(2m,\mathbb{R})$ just as in (3.2). The non-degenerate 2-form ω defines an "almost symplectic structure" invariant by the group $Sp(m,\mathbb{R})$. Note the missing "2" in our notation for this group, which is a real form of $Sp(m,\mathbb{C})$; in some sense, the complexification of **3.2** leads to the theory of hyperkähler manifolds, discussed in chapter 8.

For each group H and tensor η in **3.2**, we can investigate the Bernard structure function $T_0(p)$, giving the obstruction to the existence of a connection without torsion on the corresponding H-structure. The existence of such a torsion-free connection leads to the type of geometrical structure written above the group in question. For example, $SO(2m)$ obviously involves no additional condition; we tackle the remaining groups in the order $U(m)$, $GL(m,\mathbb{C})$, $Sp(m,\mathbb{R})$.

Any torsion-free $U(m)$-connection is unique, and exists if and only if the underlying Levi Civita connection ∇ reduces to the $U(m)$-structure P. A real $2m$-dimensional manifold is called *Kähler* if it has a torsion-free $U(m)$-structure, which implies that $\nabla\omega = 0$. Equivalently, a Kähler manifold is a Riemannian manifold whose holonomy group equals $U(m)$. The following result is a direct consequence of **2.2** and (3.5).

3.3 Lemma *The structure function $\nabla\omega$ of the $U(m)$-structure takes values in the space*

$$\Lambda^1\otimes[\![\lambda^{0,2}]\!] \;\cong\; [\![\lambda^{0,1}\otimes\lambda^{0,2}]\!]\oplus[\![\lambda^{1,2}]\!].$$

If H is one of the groups $GL(m,\mathbb{C})$ or $Sp(m,\mathbb{R})$, with Lie algebra $\mathfrak{h}$, the space

$$\operatorname{coker}\delta \;=\; \frac{\Lambda^2\otimes\Lambda^1}{\delta(\Lambda^1\otimes\mathfrak{h})} \tag{3.9}$$

containing the structure function $T_0(p)$ can be identified with some subspace of $\Lambda^1\otimes\mathfrak{u}(m)^\perp$. In each case, it follows that $T_0(p)$ is a sum of H-invariant components of $\nabla\omega$.

The spaces $\lambda^{p,q}$ are $GL(m,\mathbb{C})$-invariant, and the associated vector bundles $\lambda^{p,q}M$ consist of complexified differential forms of type (p,q) relative to the almost complex structure I. It is an easy matter to verify that the obstruction to the existence of a torsion-free $GL(m,\mathbb{C})$-connection can be identifed with the component of $\nabla\omega$ in the real space underlying

$$\begin{aligned}\lambda^{0,1}\otimes\lambda^{0,2} \;&\cong\; (\lambda^{1,0})^*\otimes\lambda^{0,2}\\ &\cong\; \mathrm{Hom}(\lambda^{1,0},\lambda^{0,2}).\end{aligned}$$

This obstruction can now be identified with the tensor determined by mapping a $(1,0)$-form $\alpha\in\Gamma(M,\lambda^{1,0}M)$ to the component $(d\alpha)^{0,2}\in\Gamma(M,\lambda^{0,2}M)$ of its exterior derivative. It vanishes if and only if

$$d(\Gamma(M,\lambda^{p,q}M)) \;\subseteq\; \Gamma(M,\lambda^{p+1,q}M\oplus\lambda^{p,q+1}M), \tag{3.10}$$

which is in turn the condition that guarantees the existence of a double complex

$$\begin{aligned} \partial\colon \Gamma(M,\lambda^{p,q}M) &\longrightarrow \Gamma(M,\lambda^{p+1,q}M),\\ \overline{\partial}\colon \Gamma(M,\lambda^{p,q}M) &\longrightarrow \Gamma(M,\lambda^{p,q+1}M), \end{aligned} \tag{3.11}$$

with $d=\partial+\overline{\partial}$ and $0=d^2=(\partial^2,\partial\overline{\partial}+\overline{\partial}\partial,\overline{\partial}^2)$.

In terms of vector fields $X,Y\in\mathfrak{X}(M)$, the $GL(n,\mathbb{C})$-structure function $T_0(p)$ translates into the so-called Nijenhuis tensor

$$\begin{aligned} N(X,Y) &= -\mathrm{Re}\,(\mathbf{1}-iI)[X+iIX,Y+iIY]\\ &= [IX,IY]-I[IX,Y]-I[X,IY]-[X,Y], \end{aligned}$$

of the almost complex structure [Fr_1], since

$$(d\alpha)^{0,2}(X,Y) \;=\; d\alpha(X+iIX,Y+iIY) \;=\; -\tfrac{1}{2}\alpha[X+iIX,Y+iIY]$$

vanishes for all $(1,0)$-forms α if and only if $N(X,Y)=0$.

Clearly N vanishes if, in a neighbourhood of each point m, there exist *closed* $(1,0)$-forms $\alpha^1,\ldots,\alpha^m$ that are linearly independent at m. This amounts to having complex coordinates

$$z^1=x^1+iy^1,\;\ldots\;,z^m=x^m+iy^m$$

on a possibly smaller neighbourhood U of m with

$$dz^i\in\Gamma(U,\lambda^{1,0}),\quad \text{or}\quad I\Big(\frac{\partial}{\partial x^r}\Big)=\frac{\partial}{\partial y^r},$$

and is precisely the condition that the $GL(m,\mathbb{C})$-structure is integrable, with local sections of the form

$$\Big\{\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^m},\frac{\partial}{\partial y^1},\ldots,\frac{\partial}{\partial y^m}\Big\}. \tag{3.12}$$

It is customary to define complex tangent vectors by the formulae

$$\begin{aligned} \frac{\partial}{\partial z^\alpha} &= \tfrac{1}{2}\Big(\frac{\partial}{\partial x^\alpha}-i\frac{\partial}{\partial y^\alpha}\Big),\\ \frac{\partial}{\partial \overline{z}^\alpha} &= \tfrac{1}{2}\Big(\frac{\partial}{\partial x^\alpha}+i\frac{\partial}{\partial y^\alpha}\Big), \end{aligned}$$

so that $\frac{\partial}{\partial z^\alpha}(dz^\beta) = \delta_{\alpha\beta}$, and

$$I(\frac{\partial}{\partial z^\alpha}) = i\frac{\partial}{\partial z^\alpha}.$$

The Newlander-Nirenberg theorem [NN] asserts that the existence of such complex coordinates is actually equivalent to the first order condition $N = 0$, which is the converse to the second statement in **1.6**. In this situation, any two sets of complex coordinates are related by a holomorphic transformation between open sets of $\mathbb{C}^m$, and M is a *complex manifold.* A Riemannian metric g on a complex manifold is called *Hermitian* if it is compatible with the complex structure in the sense of (3.8). It has components $g_{\alpha\overline{\beta}} = g(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \overline{z}^\beta})$ with respect to complex coordinates, and the associated 2-form is expressed locally as

$$\omega = -i\sum_{\alpha,\beta} g_{\alpha\overline{\beta}} dz^\alpha \wedge \overline{dz}^\beta,$$

which is the non-flat version of (3.6). The following result is well known:

3.4 Proposition *A Hermitian metric is Kähler if and only* $(d\omega)^{1,2} = 0$.

Proof. Because the Levi Civita connection ∇ is torsion-free, the exterior derivative $d\omega$ can be recovered from the components of $\nabla\omega$ in **3.3**. This is apparent from the decompositions

$$\begin{aligned} \lambda^{0,1}\otimes\lambda^{0,2} &\cong V \oplus \lambda^{0,3} \\ \lambda^{1,2} &\cong \lambda_0^{1,2} \oplus \lambda^{0,1} \end{aligned} \tag{3.13}$$

of the relevant spaces into *irreducible* $U(m)$-components; $d\omega$ is the image of $\nabla\omega$ under the anti-symmetrization

$$\Lambda^1\otimes\Lambda^2 \longrightarrow \Lambda^3,$$

whose kernel is isomorphic to the real space $[\![V]\!]$ underlying V. Observe that the component $(d\omega)^{0,3}$ of $d\omega$ of type (0,3) is a constituent of N, vanishing automatically when M is complex. On the other hand $(d\omega)^{1,2}$ determines the component of $\nabla\omega$ in **3.3** complementary to the Nijenhuis tensor. □

Using (3.9), it follows that it is precisely the $Sp(m,\mathbb{R})$-structure function $T_0(p)$ which can be identified with $d\omega$, regarded as a sum of components of $\nabla\omega$. The analogue of the Newlander-Nirenberg theorem for a $Sp(m,\mathbb{R})$-structure, namely that the structure is integrable if and only $d\omega = 0$, is the more elementary theorem of Darboux. This time, integrability is equivalent to the existence of real coordinates $x^1, y^1, \ldots, x^m, y^m$ such that $\omega = -2\sum_{\alpha=1}^m dx^\alpha \wedge dy^\alpha$. A non-degenerate closed 2-form ω on a manifold is called a *symplectic form*, and determines an element $[\omega] \in H^2(M,\mathbb{R})$ of de Rham cohomology satisfying $[\omega]^m \neq 0$ if M is compact. On a compact manifold, Moser [Mo] proved that if ω_t is a one-parameter family of symplectic forms with $[\omega_t]$ constant, then there exist diffeomorphisms F_t such that such that $\omega_t = F_t^*\omega$.

The symplectic form ω also gives rise to a homotopy class of almost complex structures on M, a representative I of which is defined by reducing the structure group to $U(m)$ as in (3.8). In these circumstances, the almost complex structure I is said to be *calibrated* by ω (a notion exploited by Harvey and Lawson in [HL]), and the resulting first Chern class $c_1(M) \in H^2(M,\mathbb{Z})$ may be compared with $[\omega]$. The homotopy class of I includes the bigger set of almost complex structures *tamed* by a fixed symplectic form ω, meaning that $\omega(IX, X)$ is positive whenever X is non-zero [Gr_1].

If M is open (i.e. $M - \partial M$ has no compact component), then any almost complex structure on M is homotopic to one calibrated by some symplectic form ω. This is a consequence of Gromov's powerful *h-principle* for open manifolds, which asserts that a section of a jet bundle representing a suitable differential relation of order r is homotopic to the r-jet of some section (see [Gr_2]). In addition it is now easy to choose ω so as to realize any class in $H^2(M,\mathbb{R})$, which contrasts with the compact case. The first example of a *compact* symplectic manifold admitting no Kähler structure was given by Thurston [T] (for descriptions from the viewpoint of Riemannian geometry, see [Ab],[CFG]). A simply-connected example was later discovered by McDuff [Mc] using the notion of symplectic blowing-up.

When M has real dimension $2m = 4$, the spaces $\lambda^{0,3}$ and $\lambda_0^{1,2}$ are zero, so in this case the complex and symplectic conditions are trully complementary. Properties of $d\omega$ on a compact complex surface were exploited by Gauduchon [Ga_1]. The general situation is illustrated below, and allows one to give a finer classification of the $U(m)$-structure P according to the non-zero components of $\nabla\omega$ [GH].

3.5 Figure Components of the $U(m)$-Structure function $\nabla\omega$, $m \geqslant 3$

$[\![V]\!]$	$[\![\lambda^{0,3}]\!]$
$[\![\lambda^{0,1}]\!]$	$[\![\lambda_0^{1,2}]\!]$

0	0
0	0

Kähler

0	0
?	?

Complex

?	0
0	0

Symplectic

Examples and further properties

Let f be a real function on $\mathbb{C}^m$. The *Levi form* of f is the real closed (1,1)-form

$$-i\partial\overline{\partial}f \;=\; -i\sum_{\alpha,\beta=1}^{m}\frac{\partial^2 f}{\partial z^\alpha \partial \overline{z}^\beta} dz^\alpha \wedge \overline{dz}^\beta,$$

and will define a Kähler metric, if it is positive definite. Conversely, a closed 2-form ω is locally expressible as $d\tau$, where $\partial(\tau^{1,0}) = 0 = \overline{\partial}(\tau^{0,1})$. Using holomorphic analogues of the Poincaré lemma, there exist functions f_1, f_2 on a sufficiently small open set such that $\tau^{1,0} = \partial f_1$ and $\tau^{0,1} = \overline{\partial} f_2$. Hence

$$\omega \;=\; \overline{\partial}\partial f_1 + \partial\overline{\partial} f_2 \;=\; -i\partial\overline{\partial} f,$$

where $f = i(-f_1 + f_2)$ is the so-called *Kähler potential.* The latter is unique, up to the addition of a *pluriharmonic function*, by definition, one whose Levi form vanishes.

The potential of the flat Kähler metric on $\mathbb{C}^{m+1}$ is the square $\|z\|^2 = \sum_{\alpha=0}^m |z^\alpha|^2$ of the usual Hermitian norm of a point $z = (z^0, \ldots, z^m)$ of $\mathbb{C}^{m+1}$. Consider instead the function $f = \log(\|z\|^2)$, whose Levi form $-i\partial\overline{\partial} f$ is unchanged when z is replaced by λz, where λ is a non-zero complex scalar which may be regarded as a coordinate on any fibre of the projection

$$\pi\colon \mathbb{C}^{m+1} - \{0\} \longrightarrow \mathbb{C}P^m$$

to the complex projective space. Since the pullback of $\partial\overline{\partial} f$ to a fibre of π is proportional to $\partial\overline{\partial}(\|\lambda\|^2) = 0$, there must exist a closed $(1,1)$-form ω on $\mathbb{C}P^m$ such that $\pi^*\omega = -i\partial\overline{\partial} f$. This 2-form is non-degenerate and defines a Kähler metric on $\mathbb{C}P^m$, the so-called *Fubini-Study metric.*

The *tautological line bundle*, denoted for the moment L^{-1}, on $\mathbb{C}P^m$ is the subbundle of the trival vector bundle $\mathbb{C}P^m \times \mathbb{C}^{m+1}$ whose fibre at $m \in \mathbb{C}P^m$ is simply the complex line containing $\pi^{-1}(m)$. Then the total space of L^{-1} is $\mathbb{C}^{m+1}$ with its origin 0 *blown up*, that is replaced by a copy of $\mathbb{C}P^m$ representing the set of directions at 0. The purely imaginary 2-form $i\omega$ may be interpreted as the curvature of a canonical connection on L^{-1} (see the beginning of chapter 8). Combining the two potentials mentioned in the preceding paragraph yields

3.6 Lemma *The 2-form $-i\partial\overline{\partial}\Big(\|z\|^2 + a\log(\|z\|^2)\Big)$ defines a Kähler potential on $\mathbb{C}^{m+1} - \{0\}$ that extends to the total space of L^{-1}, for any $a > 0$.*

The associated metric is readily seen to equal the restriction of a product metric on $\mathbb{C}P^m \times \mathbb{C}^{m+1}$. Analogous constructions apply to $\mathbb{C}P^{m+1}$ minus a point which, by projection from that point, can be identified with the total space of the dual line bundle L. The latter is the so-called *hyperplane bundle*, any section of which vanishes in turn on a hyperplane $\mathbb{C}P^{m-1}$.

The Fubini-Study metric on the complex projective space is invariant by the group $U(m+1)$ of unitary transformations of $\mathbb{C}^{m+1}$, which leads to the coset space description

$$\mathbb{C}P^m = \frac{U(m+1)}{U(m) \times U(1)} \cong \frac{SU(m+1)/\mathbb{Z}_2}{U(m)}. \tag{3.14}$$

The action of $U(m+1)$ then lifts to the total spaces of the line bundles L and $L^{-1} = L^*$, which are associated to representations of the centre of the second denominator $U(m)$. For $k \geqslant 0$, the space of holomorphic sections of the k-fold tensor product

$L^k = \bigotimes^k L$ (which, in future, we shall denote by $\mathcal{O}(k)$, which also stands for the associated sheaf of germs of local holomorphic sections) is isomorphic to the space of homogeneous polynomials of degree k, which is an irreducible representation of $U(m+1)$. In tensor notation,

$$\bigoplus_{k\geqslant 0} H^0(\mathbb{C}P^m, \mathcal{O}(k)) \cong \bigoplus_{k\geqslant 0} \bigodot^k (\mathbb{C}^{m+1})^*, \tag{3.15}$$

and when m is odd, the $k=2$ summand can be identified with the subalgebra $\mathfrak{sp}(\frac{1}{2}(m+1), \mathbb{C})$ of $\mathfrak{u}(m+1)$, a fact which will be relevant in chapter 9.

The exterior product representation $\bigwedge^k(\mathbb{C}^{m+1})$ has a similar interpretation as a space of holomorphic sections of a suitable vector bundle over $\mathbb{C}P^m$, as part of a general scheme of Beilinson [Be]. Alternatively, to exhibit it as a space of holomorphic sections of a *line* bundle, one must pass to a different base manifold, for example a complex Grassmannian, and use the Borel-Weil theorem.

If $j\colon N \to \mathbb{C}P^m$ denotes a holomorphic embedding, the pull-back $j^*\omega$ is a Kähler metric on the complex submanifold N. In other words, any projective algebraic manifold is Kähler. In the reverse direction, a compact complex manifold N admitting an integral cohomology class represented by a positive-definite $(1,1)$-form ω is called a *Hodge* manifold, and is necessarily algebraic. For Kodaira's theorem asserts that some multiple of ω represents the curvature of a holomorphic line bundle j^*L on N for which the natural mapping

$$\begin{array}{rcl} j\colon N & \longrightarrow & \mathbb{C}P^m \\ z & \mapsto & [s_0(z), \ldots, s_m(z)] \end{array} \tag{3.16}$$

is an embedding, where $\{s_0, \ldots, s_m\}$ is a basis of sections of L. Properties of j are encoded into the induced homomorphism of (3.15) onto $\bigoplus_{k\geqslant 0} H^0(N, \mathcal{O}(j^*L^k))$.

The simplest submanifolds of $\mathbb{C}P^m$ are the zero sets of a single homogeneous polynomial. The hyperquadrics correspond to degree $k=2$, a standard one being defined in homogeneous coordinates as the set $Q^{m-1} = \{\pi(z) : \sum_{\alpha=0}^m (z^\alpha)^2 = 0\}$. Equivalently, Q^{m-1} is the set of totally isotropic complex 2-dimensional subspaces of $\mathbb{C}^{m+1}$, and is isomorphic to the real Grassmannian

$$\widetilde{Gr}_2(\mathbb{R}^{m+1}) \cong \frac{SO(m+1)}{SO(m-1) \times SO(2)} \tag{3.17}$$

of oriented real 2-dimensional subspaces in $\mathbb{R}^{m+1}$.

For example, $Q^2 \cong \widetilde{Gr}_2(\mathbb{R}^4)$ is isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$, which is contained in $\mathbb{C}P^3$ via the Segre embedding determined by the tensor product $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$. The latter is the complexification of the basic representation of $SO(4)$, which is locally isomorphic to the product $SU(2) \times SU(2)$ (cf. **6.2**). A hypersurface of degree 3 in $\mathbb{C}P^3$ is a del Pezzo surface, one of degree 4 is a K3 surface (see the end of chapter 7), ones of higher degree are examples of the so-called surfaces of general type.

The subcomplex of the de Rham complex formed by the operators $\overline{\partial}$ in (3.11) is used to define the *Dolbeault cohomology*

$$H^{p,q}(M, \overline{\partial}) = \frac{\ker\,(\overline{\partial}\colon \Gamma(M, \lambda^{p,q}M) \to \Gamma(M, \lambda^{p,q+1}))}{\overline{\partial}(\Gamma(M, \lambda^{p,q-1}M)}. \tag{3.18}$$

This is best illustrated when $m = 2$, and (3.11) takes the form

$$\begin{array}{ccccc}
\Gamma(\lambda^{0,2}M) & \to & \Gamma(\lambda^{1,2}M) & \to & \Gamma(\lambda^{2,2}M) \\
\uparrow & & \uparrow & & \uparrow \\
\Gamma(\lambda^{0,1}M) & \to & \Gamma(\lambda^{1,1}M) & \to & \Gamma(\lambda^{2,1}M) \\
\uparrow \partial & & \uparrow & & \uparrow \\
\Gamma(\lambda^{0,0}M) & \stackrel{\overline{\partial}}{\to} & \Gamma(\lambda^{1,0}M) & \to & \Gamma(\lambda^{2,0}M).
\end{array}$$

The groups (3.18) appear when the double complex is derived in vertical directions, and constitute terms $E_1^{p,q} = H^{p,q}(M, \overline{\partial})$ in the associated Frölicher spectral sequence [Fr$_2$]. The residual operators ∂ induce new horizontal maps $E_1^{p,q} \to E_1^{p+1,q}$, and the resulting cohomology groups $E_2^{p,q}$ are linked by two knights' moves α, β:

$$\begin{array}{ccccc}
E_1^{0,2} & \longrightarrow & E_1^{1,2} & \longrightarrow & E_1^{2,2} \\
E_1^{0,1} & \longrightarrow & E_1^{1,1} & \longrightarrow & E_1^{2,1} \\
E_1^{0,0} & \stackrel{\partial}{\longrightarrow} & E_1^{1,0} & \longrightarrow & E_1^{2,0}
\end{array}
\qquad\qquad
\begin{array}{ccccc}
E_2^{0,2} & & E_2^{1,2} & & E_2^{2,2} \\
 & & & \beta & \\
E_2^{0,1} & & E_2^{1,1} & & E_2^{2,1} \\
 & & & \alpha & \\
E_2^{0,0} & & E_2^{1,0} & & E_2^{2,0}
\end{array}$$

This information can be used to relate de Rham and Dolbeault cohomology. For example, there is a filtration of $H^2(M, \mathbb{R})$ with successive quotients $\operatorname{coker}\alpha$, $E_2^{1,1}$, $\ker\beta$. However, when M is a *compact* complex surface, all the maps immediately above are actually zero, generalizing the fact that any holomorphic function is constant. Then the cohomological version

$$H^k(M,\mathbb{R}) = \sum_{p+q=k} H^{p,q}(M,\overline{\partial}). \tag{3.19}$$

of (3.3) is valid for $k=2$, whereas the situation for $k=1$ depends upon the parity of b_1. See, for example, the treatment in [BPV].

On the other hand, (3.19) is valid on a compact *Kähler* manifold of arbitrary dimension (cf. **4.11**). The strong compatibility between complex and Riemannian structures on a Kähler manifold M is illustrated by the following fact. Let D_p denote the horizontal space at a frame p belonging to the $U(m)$-structure P of M. Because D_p is torsion-free in the sense of **1.1**, and tangent to the $GL(m,\mathbb{C})$-structure, one can arrange an integrable section (3.12) to be tangent to D_p. With respect to the corresponding complex coordinates,

$$\begin{aligned} g_{\alpha\overline{\beta}}\Big|_m &= \delta_{\alpha\beta}, \\ \frac{\partial g_{\alpha\overline{\beta}}}{\partial z^\gamma}\bigg|_m &= 0, \end{aligned}$$

so that the metric osculates the flat one, in analogy to (2.5).

The Dolbeault complex exists only in the context of a complex structure, but it is sometimes possible to build up analogous complexes of differential operators in non-integrable situations. With this goal, it is particularly appropriate to consider almost complex manifolds M of dimension $2m=6$, for then there is the possibility that the Nijenhius tensor N actually determines an *isomorphism* $\lambda^{1,0}M \cong \lambda^{0,2}M$. The following result is the first of several we shall encounter that characterize a specific type of geometrical structure in terms of the existence of a subcomplex of the de Rham complex. In this instance, we consider the sequence

$$0 \to \Gamma(\lambda^{0,0}M) \xrightarrow{D} \Gamma(\lambda^{1,0}M \oplus \lambda^{0,1}M) \xrightarrow{D} \Gamma(\lambda_0^{1,1}M \oplus \lambda^{0,2}M) \xrightarrow{D} \Gamma(\lambda_0^{1,2}M) \to 0$$

of complex vector bundles of dimensions $1,6,11,6$, linked by successive operators D, each of which denotes exterior differentiation d followed by a suitable projection induced from the linear algebra.

3.7 Proposition *The $U(3)$-structure function $\nabla\omega$ takes values in the diagonal space $[\![\lambda^{0,1} \oplus \lambda^{0,3}]\!]$ of* **3.5** *if and only if $D^2 = 0$.*

Proof. The first two D's coincide with d, so their composition is obviously zero. It is zero for the last two if and only if the image by d of the subbundle of $(\Lambda^2 T^*M)_{\mathbb{C}}$ spanned by ω and $\lambda^{2,0}M$ (the missing ingredients) has no component in $\lambda_0^{1,2}M$. It follows that $D^2 = 0$ is equivalent to (i) $(d\omega)_0^{1,2} = 0$, and (ii) $\nabla\omega$ having no component in the subspace V of $\mathrm{Hom}(\lambda^{2,0}, \lambda_0^{1,2})$. $\boxed{\text{f}}$

Other features pertinent to the above situation were studied by O'Brian and Rawnsley [OR]. The more restrictive condition $\nabla\omega \in [\![\lambda^{0,3}]\!]$ characterizes the class of *nearly Kähler* manifolds, which includes the sphere S^6, whose Riemannian metric is determined up to homothety by the Nijenhuis tensor of a canonical non-integrable almost complex structure. The importance of this class can be seen in a theorem of Gray [G_3, theorem 5.2], which asserts that any 6-dimensional nearly Kähler manifold, which is not Kähler, is necessarily Einstein. Further examples arise as total spaces of S^2-bundles over 4-manifolds, and will play an important role in our construction of metrics with exceptional holonomy.

4 Riemannian and Kähler Curvature

The curvature tensor R of a Riemannian manifold can be defined directly by means of various derivatives of the metric g. For a surface it is just a scalar, the so-called Gaussian curvature, but in higher dimensions it lies in a space $\mathfrak{R}$ which is a non-trivial representation of the orthogonal group $O(n)$, and whose properties are described by well-known symmetries of R. The irreducible components of $\mathfrak{R}$ are determined as an informal exercise in representation theory; this leads to the definition of the Weyl, Ricci and scalar curvature. Each of these components has relevance to particular geometrical questions, but our main interest lies in what happens to them in the presence of special holonomy.

The space $\mathfrak{R}^H$ of curvature tensors associated to a holonomy group H takes into account restrictions imposed on R by the holonomy reduction, which are often severe. In due course, the structure of $\mathfrak{R}^H$ will be found for a handful of Lie groups; in the present chapter we confine ourselves to the subgroups $U(m)$ and $SU(m)$ of $SO(2m)$ that arise in connection with Kähler geometry. Curvature plays an important role in the study of differential operators between vector bundles, via the Weitzenbock formula, which is essentially a comparison of two different ways of decomposing a double covariant derivative. Hodge theory can then provide valuable information about the cohomology groups of the manifold.

Spaces of curvature tensors

Let M be a Riemannian manifold of dimension $n \geqslant 3$. We shall consider the value of the Riemann curvature tensor $R = R(p)$ with respect to a fixed orthonormal frame p. The exterior derivative of the vanishing torsion equation $\Theta = 0$ on the principal $O(n)$-bundle P of orthonormal frames yields the first Bianchi identity $[\Phi, \theta] = 0$, valid for any connection without torsion. As in chapter 1, brackets denote wedging combined with the action of the Lie algebra $\mathfrak{so}(n)$ on $\mathbb{R}^n$. To obtain the more familiar version of the identity, one must therefore let the curvature operator R_{xy} act on z, and then anti-symmetrize the vectors $x, y, z \in \mathbb{R}^n$. Because R_{xy} is already anti-symmetric in x, y, a cyclic sum is sufficient:

4.1 Proposition $R_{xy}z + R_{yz}x + R_{zx}y = 0$.

An element $h^{-1}\in O(n)$ acts on $R=R(p)$ by

$$\begin{aligned}(h^{-1}R)_{xy} &= R(ph)_{xy}\\ &= (\operatorname{Ad}h^{-1})R_{hx,hy}\\ &= h^{-1}\circ R_{hx,hy}\circ h,\end{aligned}$$

and differentiation determines the action of the Lie algebra:

$$(AR)_{xy}=R_{xy}\circ A-A\circ R_{xy}+R_{Ax,y}+R_{x,Ay},\quad A\in\mathfrak{so}(n). \tag{4.1}$$

For a fuller understanding of the action of the group on the space of curvature tensors, it helps to "lower the first index" of R according to the isomorphism

$$\Lambda^2\otimes\mathfrak{so}(n)\ \cong\ \Lambda^2\otimes\Lambda^2 \tag{4.2}$$

(cf. (2.2)), and **4.1** can now be expressed by saying that R lies in the kernel of the composition

$$b\colon\Lambda^2\otimes\Lambda^2\hookrightarrow\Lambda^2\otimes\Lambda^1\otimes\Lambda^1\to\Lambda^3\otimes\Lambda^1,$$

consisting in essence of anti-symmetrization on the first three factors. Accordingly, the full space of curvature tensors associated to the orthogonal group $O(n)$ is defined by

$$\mathfrak{R}\ =\ \mathfrak{R}^{O(n)}\ =\ \ker b. \tag{4.3}$$

Related to b is the linear mapping

$$a\colon\Lambda^2\otimes\Lambda^2\longrightarrow\Lambda^4,$$

determined by wedging together 2-forms. Since this is a symmetric pursuit, the kernel of a contains the skew part $\bigwedge^2(\Lambda^2)$ of the tensor product, and there is an induced map $\bigodot^2(\Lambda^2)\to\Lambda^4$. Incidentally a 2-form σ satisfies the Plücker equation $a(\sigma\otimes\sigma)=0$ if and only if $\sigma=\alpha\wedge\beta$ is a simple or indecomposable form. This useful property is readily checked by observing that an arbitrary 2-form can be written in the form

$$\sigma\ =\ \sum_{i=1}^{n-1}c_ie^i\wedge e^{i+1},$$

for some orthonormal basis $\{e^1,\dots,e^n\}$ of $\Lambda^1=(\mathbb{R}^n)^*$, with $c_i\in\mathbb{R}$.

4.2 Lemma $\mathfrak{R} = \ker a \cap \bigodot^2(\Lambda^2)$.

Proof. With the task of identifying $\ker b$, we find ourselves in a similar, but more complicated, situation to (2.3). As in that case it is possible to give an explicit proof (an elegant one is furnished by the tetrahedron in [Mi]) of the required result. However, we shall require more precise information, and a more systematic approach is to decompose all spaces in sight into irreducible components relative to the action of $O(n)$.

It is not hard to guess that there are three irreducible components of $\Lambda^1 \otimes \Lambda^3$ under the action of $O(n)$, at least for $n \geqslant 4$. There is the $GL(n, \mathbb{R})$-invariant subspace of totally skew forms, which we can denote by Λ^4 without ambiguity. There is also the subspace $c^*(\Lambda^2)$, where c^* is the adjoint of the $O(n)$-equivariant contraction $\Lambda^1 \otimes \Lambda^3 \to \Lambda^2$. Now, it is well known that Λ^k is $O(n)$-irreducible, for all k. Furthermore, general principles (which we shall elaborate later) then affirm that the orthogonal complement U of $\Lambda^4 \oplus c^*(\Lambda^2)$ in $\Lambda^1 \otimes \Lambda^3$ is also irreducible. Proceeding in this fashion, we obtain the irreducible decompositions

$$\begin{aligned} \Lambda^1 \otimes \Lambda^3 &\cong \Lambda^2 \oplus \Lambda^4 \oplus U, \\ \bigwedge^2(\Lambda^2) &\cong \Lambda^2 \oplus V, \\ \bigodot^2(\Lambda^2) &\cong \mathbb{R} \oplus \Sigma^2_0 \oplus \Lambda^4 \oplus W. \end{aligned} \tag{4.4}$$

The projection $\bigwedge^2(\Lambda^2) \to \Lambda^2$ may be identifed, via (2.2), with the Lie bracket. When $n = 3$, the above decompositions remain valid provided Λ^2, U, V, W are all replaced by zero.

The linear map b is defined by symmetries, so is certainly $O(n)$-invariant. By I. Schur's lemma, its restriction to each summand of $\Lambda^2 \otimes \Lambda^2$ is zero or an isomorphism. It is an easy matter to check that the image of b does not miss any summand of $\Lambda^1 \otimes \Lambda^3$, and consequently that b is onto. It follows from a dimension count that $U \cong V \ncong W$, and b has to be zero on the irreducible spaces W, Σ^2_0, $\mathbb{R}$, since these do not occur in $\Lambda^1 \otimes \Lambda^3$. The lemma follows. □

As a corollary of the above proof, we record for future purposes

4.3 Theorem [ST] *At any $p \in P$, $R(p)$ belongs to the space*

$$\mathfrak{R} \cong \begin{cases} \Sigma^2_0 \oplus \mathbb{R}, & n = 3 \\ W \oplus \Sigma^2_0 \oplus \mathbb{R}, & n \geqslant 4. \end{cases}$$

The Levi Civita connection, and therefore the curvature tensor $R = R(p)$, are determined explicitly by the Riemannian metric. Using coordinates for which the metric is constant to first order at $m = \pi(p)$, so that $\partial g_{ij}/\partial x^k|_m = 0$, we find

$$R_{ijkl} = \frac{1}{2}\left[\frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}\right]_m .$$

It is this expression that emphasizes the inherent *symmetries* of R, as opposed to its *skew*-symmetries. Indeed, the tensor with coefficients

$$S_{ikjl} = \tfrac{1}{2}(R_{ijkl} + R_{kjil}), \tag{4.5}$$

is symmetric, not only in the indices i, k, but also in j, l, and is also unchanged when these two pairs are interchanged. Moreover it belongs to the kernel of the total symmetrization

$$s\colon \Sigma^2 \otimes \Sigma^2 \longrightarrow \Sigma^4.$$

As a reflection of a widespread, and often mysterious, duality between the symmetric and exterior algebras, one has the following analogue of **4.2**:

4.4 Proposition *Equation* (4.5) *induces an isomorphism of* $\mathfrak{R}$ *with the space* $\ker s \cap \bigodot^2(\Sigma^2)$.

This and related decompositions may be found, for example, in [BBG].

The value of the curvature tensor with reference to the frame p is completely determined by polarizing $S(x, y) = g(R_{xy}x, y)$ separately in x and y. In fact, the *sectional curvature*

$$\frac{S(x, y)}{\|x \wedge y\|^2} = \frac{g(R_{xy}x, y)}{\|x\|^2\|y\|^2 - (g(x, y))^2} \tag{4.6}$$

of the 2-plane spanned by the tangent vectors px, py represents the induced Gaussian curvature of the corresponding surface generated by the exponential map. If the value c of (4.6) depends only on m, then S is the orthogonal projection of $g \otimes g$ in $\ker s$, and R has coefficients

$$R_{ijkl} = c(g_{ik}g_{jl} - g_{il}g_{jk}), \tag{4.7}$$

and belongs to the 1-dimensional invariant subspace of $\mathfrak{R}$. If (4.7) also holds for all $m \in M$, an application of the second Bianchi identity **10.1** establishes F. Schur's lemma, namely that c is even independent of m (assuming always that $n \geqslant 3$).

Although **4.4** provides a more elegant description of R than **4.2**, it is in reality more complicated, since the space $\Sigma^2 = \Sigma^2_0 \oplus \mathbb{R}$ is reducible. Indeed, performing the trace $\Sigma^2 \to \mathbb{R}$ on either factor gives the *Ricci tensor*

$$Ric_{jl} = g^{ik} S_{ikjl} = g^{ik} R_{ijkl} \tag{4.8}$$

that encapsulates the components Σ^2_0 and $\mathbb{R}$ of $\mathfrak{R}$. Performing the remaining trace gives the *scalar curvature*

$$t \;=\; g^{ik} g^{jl} R_{ijkl}.$$

The Ricci tensor is remarkable in that it is exactly the same species of tensor as the Riemannian metric g, namely a quadratic form on each tangent space. Moreover, Ric is unchanged when g is rescaled by some constant. A Riemannian manifold M of dimension $n \geqslant 3$ is called *Einstein* if the Ricci tensor is a constant multiple of the metric at each point, so that R has no component in the component Σ^2_0 of $\mathfrak{R}$. Then

$$Ric_{jl} \;=\; \frac{1}{n} t \, g_{jl}, \tag{4.9}$$

and another application of **10.1** shows that the scalar curvature t must actually be constant on M. In the special case $n = 3$, it is clear from **4.3** that (4.7) and (4.9) must be equivalent.

The component of R in the space W is called the *Weyl* tensor. It is only present when $n \geqslant 4$, in which case it is known to be the obstruction to the integrability of the underlying conformal structure of M, defined by the group $CO(n) = \mathbb{R}^+ \times O(n)$ [Ei]. A typical representation-theoretic argument is illustrated by the simpler

4.5 Lemma *The component of R in W (regarded as a tensor with values in $\Lambda^2 \otimes \mathfrak{so}(n)$) is conformally invariant.*

Proof. If the Riemannian metric g is altered by a conformal factor to $g' = e^f g$ then, by **1.7**, the tensor ξ measuring the difference of the corresponding Levi Civita connection forms has values in the kernel of

$$\delta \colon \Lambda^1 \otimes \mathfrak{co}(n) \longrightarrow \Lambda^2 \otimes \Lambda^1,$$

where $\mathfrak{co}(n) = \mathbb{R} \oplus \mathfrak{so}(n)$. In view of (2.3), this kernel is isomorphic to Λ^1; indeed ξ is determined by df. From

$$\begin{aligned} \Phi' - \Phi \;&=\; d\phi' - d\phi + \tfrac{1}{2}[\phi', \phi'] - \tfrac{1}{2}[\phi, \phi] \\ &=\; d(\theta\xi) + [\phi, \theta\xi] + \tfrac{1}{2}[\theta\xi, \theta\xi] \\ &=\; (\theta \wedge \theta)(\nabla\xi + \tfrac{1}{2}[\xi, \xi]), \end{aligned}$$

it follows that $R' - R$ has values in the $O(n)$-module $\Lambda^1 \otimes \Lambda^1$, and so belongs to $\mathbb{R} \oplus \Sigma_0^2$. $\square$

Curvature of a Kähler manifold

For the remainder of this section, we continue to fix an orthonormal frame $p \in P$. Let $Q = Q(p)$ denote the holonomy bundle consisting of frames obtained from p by parallel transportation, and let $H = H(p)$ be the corresponding holonomy group. On Q the curvature operator takes values in the Lie algebra $\mathfrak{h}$ of H; combining this fact with **4.2** yields

4.6 Proposition *At any point $q \in Q$, the curvature $R(q)$ belongs to the space*

$$\mathfrak{R}^H = \mathfrak{R} \cap (\mathfrak{h} \otimes \Lambda^2) = \ker a \cap \textstyle\bigodot^2 \mathfrak{h}.$$

We shall refer to $\mathfrak{R}^H$ as the *reduced space of curvature tensors* corresponding to the subgroup H of $O(n)$. The label H will remind us to treat it as a representation of the Lie group, rather than just the Lie algebra, although the notation is abusive since the definition depends crucially on the way H acts on $O(n)$, and not just H as an abstract group:

$$\begin{array}{ccccc} \mathfrak{R} & \hookrightarrow & \bigodot^2(\Lambda^2) & \xrightarrow{a} & \Lambda^4 \\ \cup & & \cup & & \| \\ \mathfrak{R}^H & \hookrightarrow & \bigodot^2 \mathfrak{h} & \longrightarrow & \Lambda^4. \end{array} \qquad (4.10)$$

Clearly, a knowledge of irreducible components of $\mathfrak{R}^H$ is essential to an understanding of the holonomy reduction, but it also provides a tool for the classification of possible holonomy groups. For many subgroups $H \subset O(n)$, the restriction of a to $\bigodot^2 \mathfrak{h}$ is injective, so $\mathfrak{R}^H = 0$, and H is ruled out of court.

The most important case is that in which H equals the group $U(m)$ of unitary transformations of a complex vector space $\lambda^{1,0}$. As usual, the underlying real vector space of dimension $n = 2m$ is denoted by Λ^1, and from (3.5), we have

$$\begin{aligned} \textstyle\bigodot^2 \mathfrak{h} &\cong \textstyle\bigodot^2(\lambda^{1,1}) \\ &\cong \textstyle\bigodot^2(\lambda^{1,0}) \otimes \bigodot^2(\lambda^{0,1}) \oplus \bigwedge^2(\lambda^{1,0}) \otimes \bigwedge^2(\lambda^{0,1}) \\ &\cong \sigma^{2,2} \oplus \lambda^{2,2}. \end{aligned}$$

In analogy to the decomposition

$$\lambda^{2,2} \cong \lambda_0^{2,2} \oplus \lambda_0^{1,1} \oplus \mathbb{R}, \tag{4.11}$$

which is only really valid when $m \geqslant 4$, we may write

$$\sigma^{2,2} \cong B \oplus \lambda_0^{1,1} \oplus \mathbb{R}, \tag{4.12}$$

for all $m \geqslant 2$, where B denotes a "primitive" component.

4.7 Proposition [Al$_1$] *For $m \geqslant 2$,*

$$\begin{aligned} \mathfrak{R}^{U(m)} &\cong B \oplus \lambda_0^{1,1} \oplus \mathbb{R}, \\ \mathfrak{R}^{SU(m)} &\cong B. \end{aligned}$$

Proof. At this point, it is possible to make an informed guess as to what effect the map $a\colon \bigodot^2 \mathfrak{h} \to \Lambda^4$ has on the summands of (4.11) and (4.12). As in the discussion of $O(n)$-invariance, Schur's lemma tells us that the restriction of a to each of these irreducible components is either zero or an isomorphism. First suppose $m \geqslant 4$; since

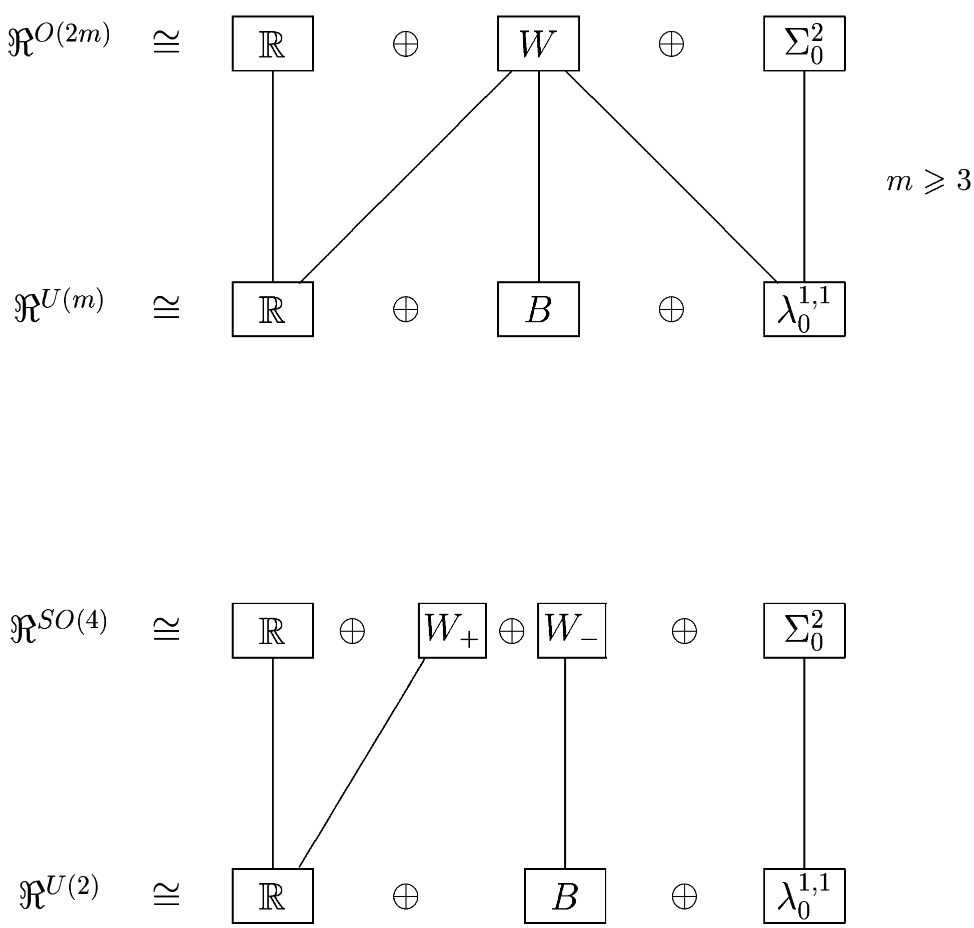
$$\Lambda^4 \cong [\![\lambda^{4,0}]\!] \oplus [\![\lambda_0^{3,1}]\!] \oplus [\![\lambda^{2,0}]\!] \oplus [\lambda_0^{2,2}] \oplus [\lambda_0^{1,1}] \oplus \mathbb{R}$$

(cf. (3.7)) does not contain B, the latter is forced to put in an appearance in the kernel of a. Furthermore, as a general guideline, one expects the image of a to be as large as Schur's lemma allows; in this case that means equal to the subspace $[\lambda^{2,2}]$ of Λ^4. Verification of this last assertion is easy. If $\{e^1, \ldots, e^m\}$ is a unitary basis of $\lambda^{1,0}$, then the symmetric product $(e^1 \wedge \overline{e}^2) \odot (e^3 \wedge \overline{e}^4)$ defines a complex element of $\lambda_0^{2,2}$, which a maps to $e^1 \wedge e^3 \wedge \overline{e}^4 \wedge \overline{e}^2$. Similarly, the image of $(e^1 \wedge \overline{e}^2) \odot (e^1 \wedge \overline{e}^2)$ by a has non-zero components in both $\lambda_0^{1,1}$ and $\mathbb{R}$. The cases $m = 2, 3$ are similar, except for the disappearance of certain summands in the above decompositions.

The Lie algebra of the group $SU(m)$ of *special* unitary transformations of $\lambda^{1,0}$ is obtained by removing the centre from $\mathfrak{u}(m)$, so $\mathfrak{su}(m) \cong \lambda_0^{1,1}$. Consequently, $\bigodot^2(\lambda_0^{1,1})$ is obtained by removing $\lambda_0^{1,1} \oplus \mathbb{R}$ from the components of $\bigodot^2(\lambda^{1,1})$, and only B remains in the kernel of a. □

This proof supplies more than the irreducible components of the spaces of curvature tensors; one can also deduce how those components behave under the inclusion $\mathfrak{R}^{U(m)} \hookrightarrow \mathfrak{R}$. In the following diagrams, lines represent non-zero projections that couple the respective components of the spaces of curvature tensors.

4.8 Figure Riemannian versus Kähler curvature



The component of R in the space B is called the *Bochner tensor*, and plays a role in Kähler geometry somewhat analogous to that of the Weyl tensor in Riemannian geometry [Y]. For generalizations to the non-Kähler situation, consult [TV]. The case of a real 4-dimensional Kähler manifold requires some comment, for then the image of B in $\mathfrak{R}$ is actually an irreducible representation of the *special* orthogonal group $SO(4)$. Indeed, when $n = 4$, neither of the spaces W, Λ^2 are irreducible by $SO(4)$, a fact that develops into the theory of self-duality (see chapter 7). Thus, to some extent the Kähler decomposition is already detected at the Riemannian level; in particular, the Bochner tensor coincides with exactly one half of the Weyl tensor.

The Ricci tensor (4.8) of a Kähler manifold M is determined by the components of R in the submodules $\mathbb{R}$ and $\lambda_0^{1,1}$ of $\mathfrak{R}^{U(m)}$. Because $SU(m)$ is the kernel of the representation

$$\det : U(m) \longrightarrow U(1) \subset \mathrm{Aut}(\lambda^{m,0}), \tag{4.13}$$

a holonomy reduction to $SU(m)$ is characterized by the existence, on any simply-connected domain, of a covariant constant section of the associated *canonical line bundle* $\kappa = \lambda^{m,0}M$. In fact, *Ric* can also be identified with the $(1,1)$-form $\mathrm{tr}(R)$ which represents the curvature of κ, where this time the trace

$$\mathrm{tr} : \lambda^{1,1}\otimes\mathfrak{u}(m) \longrightarrow \lambda^{1,1}\otimes\mathrm{End}(\lambda^{m,0}) \cong \lambda^{1,1}$$

is induced from the derivative of (4.13). Figure **4.8** indicates that $\mathrm{tr}(R)$ is completely determined by the Weyl tensor of M, provided $m \geqslant 3$.

4.9 Corollary *(i) The Ricci tensor of a Kähler manifold vanishes if and only if its restricted holonomy group H^0 is contained in $SU(m)$, $m \geqslant 2$;*
(ii) A conformally flat Kähler manifold of real dimension at least six is flat.

A counter-example to (ii) with $m = 2$ is provided by the product of two real 2-dimensional surfaces with equal and opposite constant Gaussian curvatures.

With respect to local complex coordinates, the curvature tensor R of a Kähler manifold is completely determined by its components

$$R_{\alpha\overline{\beta}\gamma\overline{\delta}} = g\Big(R_{\frac{\partial}{\partial z^\alpha},\frac{\partial}{\partial \overline{z}^\beta}}\frac{\partial}{\partial z^\gamma},\frac{\partial}{\partial \overline{z}^\delta}\Big),$$

and the Ricci form is

$$\mathrm{tr}(R) = -i\sum_{\gamma,\delta} R_{\gamma\overline{\delta}}dz^\gamma \wedge \overline{dz}^\delta, \tag{4.14}$$

where $R_{\gamma\overline{\delta}} = g^{\alpha\overline{\beta}}R_{\alpha\overline{\beta}\gamma\overline{\delta}} = g^{\alpha\overline{\beta}}R_{\alpha\overline{\delta}\gamma\overline{\beta}}$ is the version of (4.8) in complex coordinates. In analogy to (4.7), the invariant 1-dimensional subspace of $\Re^{U(m)}$ is spanned by a tensor with components

$$R_{\alpha\overline{\beta}\gamma\overline{\delta}} = g_{\alpha\overline{\beta}}g_{\gamma\overline{\delta}} + g_{\alpha\overline{\delta}}g_{\beta\overline{\gamma}}, \tag{4.15}$$

that has constant *holomorphic sectional curvature* $S(x, Ix) = g(R_{x,Ix}x, Ix) = -2$.

The Weitzenbock formula

Let M be a Riemannian manifold with holonomy bundle $Q = Q(p)$ and group $H = H(p)$. Let V be some representation of H, and α an equivariant V-valued function on Q, or equivalently a section of the associated vector bundle $VM = Q \times_H V$. So far, we have only really considered the curvature as a tensor acting on the tangent bundle. In general, one of the Lie algebra factors of the curvature

$$R = R(p) \in \bigodot^2\mathfrak{h} \subset \Lambda^2\otimes\mathfrak{h}$$

may be applied to α to produce a tensor

$$R\alpha \in \mathfrak{h}\otimes V \subset \Lambda^2\otimes V. \tag{4.16}$$

According to the Ricci identity **1.2**, this equals the image of the iterated covariant derivative $\nabla^2\alpha$ under the algebraic skewing map $\Lambda^1\otimes\Lambda^1\otimes V \longrightarrow \Lambda^2\otimes V$. One can also contract (4.16) further by applying the residual $\mathfrak{h}$ to V to produce

$$\widetilde{R}\alpha \in V, \tag{4.17}$$

where $\widetilde{R}$ should be thought of as the image of R under composition $\bigodot^2\mathfrak{h} \to \operatorname{End} V$ of endomorphisms of V.

As an illustration, we first take $V = \Lambda^k$ to be the space of k-forms, so that $\nabla\alpha$, which belongs to

$$\Lambda^1\otimes\Lambda^k \cong \Lambda^{k+1} \oplus U^k \oplus \Lambda^{k-1}.$$

Here U^k is the $O(n)$-irreducible intersection of the kernels of the obvious skewing and contraction linear mappings (cf. (4.4)). Each space on the right-hand side determines a first order differential operator; the component of $\nabla\alpha$ in Λ^{k+1} equals the exterior derivative $d\alpha$ (cf. **1.4**), and the component in Λ^{k-1} is by the definition *codifferential* $d^*\alpha$. When $k=1$, the component of $\nabla\alpha$ in $U^2 = \Sigma^2_0$ vanishes if and only if α is dual to a vector field X preserving the conformal structure; the stronger assertion that $\nabla\alpha$ is totally skew is equivalent to X being a Killing vector field (cf. (5.4)).

Now $\nabla^2\alpha$ belongs to

$$\Lambda^1\otimes(\Lambda^1\otimes\Lambda^k) \;=\; (\Lambda^1\otimes\Lambda^1)\otimes\Lambda^k,$$

inside which we may identify the following components:

$$\left.\begin{array}{rcl} \nabla d\alpha & \in & \Lambda^1\otimes\Lambda^{k+1} \\ \nabla d^*\alpha & \in & \Lambda^1\otimes\Lambda^{k-1} \\ & & \Lambda^1\otimes U^k \end{array}\right| \begin{array}{rcl} \Lambda^2\otimes\Lambda^k & \ni & R\alpha \\ \mathbb{R}\otimes\Lambda^k & \ni & \nabla^*\nabla\alpha \\ \Sigma^2_0\otimes\Lambda^k. & & \end{array} \tag{4.18}$$

The second component on the right is effectively a k-form, obtained by taking the trace of $\nabla^2\alpha$, or equivalently applying an appropriately-defined operator ∇^* to $\nabla\alpha$. Each of the six spaces in (4.18) contains a unique submodule isomorphic to $V=\Lambda^k$. The components of $\nabla d\alpha$, $\nabla d^*\alpha$ in these submodules equal $d^*d\alpha$, $dd^*\alpha$ respectively, and it is easy to see that their sum has zero component in the trace-free space $\Sigma^2_0\otimes\Lambda^k$. This leads to the Weitzenbock formula:

4.10 Proposition *The Laplacian* $\Delta = dd^* + d^*d$ *satisfies*

$$\Delta\alpha = \nabla^*\nabla\alpha - 2\widetilde{R}\alpha.$$

When M is compact, d^* and ∇^* represent formal adjoints with respect to the global inner product

$$\langle\alpha,\beta\rangle \;=\; \int_M g(\alpha,\beta)\vartheta$$

on the space $\Omega^kM = \Gamma(M,\Lambda^kT^*M)$ of smooth k-forms (ϑ denotes the volume form on M, and they are "formal" because Ω^kM is not a Hilbert space). Thus

$$\begin{aligned}\langle\Delta\alpha,\alpha\rangle \;&=\; \|d^*\alpha\|^2 + \|d\alpha\|^2,\\ \langle\nabla^*\nabla\alpha,\alpha\rangle \;&=\; \|\nabla\alpha\|^2,\end{aligned}$$

and we may now consider the space

$$\begin{aligned}\mathcal{H}^k &= \{\alpha\in\Omega^k M : \Delta\alpha = 0\}\\ &= \{\alpha\in\Omega^k M : d\alpha = 0 = d^*\alpha\}\end{aligned}$$

of harmonic forms. The main conclusions of Hodge theory then assert that $\mathcal{H}^k$ is finite-dimensional, that there is an orthogonal decomposition

$$\begin{aligned}\Omega^k M &= \mathcal{H}^k\oplus\Delta(\Omega^k M)\\ &= \mathcal{H}^k\oplus d(d^*(\Omega^k M))\oplus d^*(d(\Omega^k M))\\ &= \mathcal{H}^k\oplus d(\Omega^{k-1}M)\oplus d^*(\Omega^{k+1}M),\end{aligned}$$

and therefore that $\mathcal{H}^k$ is isomorphic to the de Rham cohomology group $H^k(M,\mathbb{R})$.

If $\widetilde{R}$ is zero, $\Delta\alpha = 0$ implies $\nabla\alpha = 0$. Thus any harmonic form α is necessarily covariant constant, and by **1.3** the holonomy group H must lie in the stabilizer of α. For example, when $k = 1$, it is clear that $\widetilde{R}\in\mathrm{End}(\Lambda^1)$ can involve only the Ricci tensor Ric, so a compact irreducible Riemannian manifold with $Ric = 0$ has $b_1 = 0$. These techniques where pioneered by Bochner [Bo],[BY], and furnish vanishing theorems under the hypothesis that $\left\langle\widetilde{R}\alpha,\alpha\right\rangle < 0$ for all non-zero α, which generally means that the curvature is positive in an appropriate sense.

A second class of examples involves the representation $\lambda^{p,0}$ defining the bundle of forms of type $(p,0)$ on a Kähler manifold M; since the submodule B defining the Bochner tensor does not appear in

$$\mathrm{End}\,V = (\lambda^{p,0})^*\otimes\lambda^{p,0}\cong\lambda^{p,p},$$

once again the operator $\widetilde{R}$ applied to the bundle $\lambda^{p,0}M$ involves only the Ricci tensor. From this may be deduced the fact that a compact Kähler manifold with positive definite Ricci form has $b^{p,0} = 0$, for all $p\geqslant 1$ [K$_2$]. Next, consider the representation

$$V = \bigoplus_{p=0}^{m}\lambda^{p,0}\otimes(\lambda^{m,0})^{-\frac{1}{2}},$$

of $SU(m)\times U(1)$. The choice of the exponent of $\lambda^{m,0}$ ensures both that $V\cong\overline{V}$, and that the action of $\widetilde{R}$ on V is determined solely by the scalar curvature t. In fact, V extends to a representation of $Spin(2m)$ (see chapter 12), which leads in general to Lichnerowicz's vanishing theorem [L$_2$], which asserts that a compact Riemannian spin manifold with $t > 0$ possesses no non-zero harmonic spinors.

Since both the "rough Laplacian" $\nabla^*\nabla$ and the curvature respect any holonomy reduction, we may conclude that any decomposition of $\bigoplus_{k=0}^{n} \Lambda^k$ into H-invariant submodules induces a corresponding decomposition of the cohomology. Because $\widetilde{R}$ depends only on the choice of representation, one may also deduce that isomorphic submodules, in whatever dimension, correspond to isomorphic cohomology spaces. For example, (3.3) and **3.1** imply the Lefschetz decomposition:

4.11 Theorem *On a compact Kähler manifold M of real dimension $2m$, there is a decomposition of cohomology*

$$H^k(M,\mathbb{R}) \cong \bigoplus_{p+q=k} \bigoplus_{r=0}^{\min(p,q)} H_0^{p-r,q-r}(M), \quad 0 \leqslant k \leqslant m.$$

The even Betti numbers of any compact symplectic manifold are all non-zero because of the presence of the closed forms ω^k. A modest consequence of **4.11** is that the odd Betti numbers of M are even; this is in general false in the more general symplectic setting. Further consequences of the existence of a Kähler metric on a compact manifold exploit the basic identity

$$\Delta = 2\Delta_\partial = 2\Delta_{\overline{\partial}},$$

relating the Laplacians of the operators $d, \partial, \overline{\partial}$ [We]. This can be used to prove that a differential form α satisfying $\partial d\alpha = 0 = \overline{\partial} d\alpha$ equals $\partial\overline{\partial}\beta$ for some β, which in turn implies the degeneration of the Frölicher spectral sequence. These last statements are also valid on compact complex manifolds which can be blown up to Kähler manifolds, and were used in [DGMS] to deduce, in particular, the vanishing of all Massey products computed from differential forms.

It was Chern [C] who first proposed the systematic generalization of the above techniques to other holonomy groups, which we shall discuss later.

5 Lie Algebras and Symmetric Spaces

The previous chapter included a determination of the space of curvature tensors for the holonomy groups $O(n)$, $U(m)$ and $SU(m)$. In the first two cases, an invariant tensor corresponds to the curvature of some model space; more generally an invariant element R of $\mathfrak{R}^H$ determines the Lie algebra $\mathfrak{g}$ of isometries of some homogeneous Riemannian manifold, called a symmetric space, whose holonomy group is H. The classification of these spaces, first carried out by É. Cartan, supplies quite a long list of holonomy groups. Inherent in this classification is the notion of duality and a remarkable correspondence between irreducible symmetric spaces and simple Lie algebras, which had themselves been classified in [C_1].

Following a discussion of the curvature tensor of symmetric spaces, we show that manifolds with special holonomy reduction are often equipped with a closed 4-form, a fact that makes them amenable to techniques generalizing those of symplectic geometry. Homogeneous spaces then provide a setting for the theory of connections, and in this context symmetric spaces are defined by the vanishing of torsion. Basic inclusions between classical Lie groups, invaluable for the sequel, provide examples of symmetric spaces, as do compact Lie groups. Mention is also made of Hermitian symmetric spaces, which are characterized by the existence of an invariant complex structure.

The Cartan algebra

Ignoring geometrical applications for the moment, we first consider, for any Lie subgroup H of $O(n)$, the space $\mathfrak{R}^H$ of curvature tensors defined in **4.6**.

5.1 Proposition *If R is an H-invariant element of $\mathfrak{R}^H$, then $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}^n$ has the structure of a Lie algebra with $[x, y] = R_{xy}$ for $x, y \in \mathbb{R}^n$.*

Proof. With some analogy to (1.11), the Lie bracket of $\mathfrak{h}$ is extended to a skew-symmetric operation on $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}^n$ by setting

$$[A, x] = -[x, A] = Ax,$$

$$[x, y] = R_{xy}, \qquad A \in \mathfrak{h},\ x, y \in \mathbb{R}^n.$$

This definition converts the Bianchi identity **4.1** into

$$[[x,y],z]+[[y,z],x]+[[z,x],y]=0,$$

which is part of the Jacobi identity (1.1) for $\mathfrak{g}$. The $\mathfrak{h}$-invariance of R, combined with (4.1) gives

$$0 \;=\; (AR)_{xy} \;=\; [R_{xy},A]+R_{Ax,y}+R_{x,Ay},$$

or equivalently,

$$[[x,y],A]+[[y,A],x]+[[A,x],y]=0.$$

Because $\mathfrak{h}$ is deemed to be a subalgebra of $\mathfrak{g}$, the Jacobi identity holds when all three elements belong to $\mathfrak{h}$. The remaining part involves two elements A,B in $\mathfrak{h}$, and one element x in $\mathbb{R}^n$, and is a rearrangement of the formula

$$[A,B]x \;=\; A(Bx)-B(Ax),$$

expressing the Lie algebra representation of $\mathfrak{h}$ on $\mathbb{R}^n$. □

It is customary to indicate the complement $\mathbb{R}^n$ of $\mathfrak{h}$ in $\mathfrak{g}$ by $\mathfrak{m}$. The Lie bracket relations

$$[\mathfrak{h},\mathfrak{h}]\subseteq\mathfrak{h},\quad [\mathfrak{h},\mathfrak{m}]\subseteq\mathfrak{m},\quad [\mathfrak{m},\mathfrak{m}]\subseteq\mathfrak{h} \tag{5.1}$$

are then characterized by the existence of a Lie algebra automorphism σ of $\mathfrak{g}$, whose restriction to $\mathfrak{h}$ is the identity, and to $\mathfrak{m}$ minus the identity. A Lie algebra $\mathfrak{g}$ carrying such an automorphism is called *involutive* or *symmetric*. In addition, the assumption that H is a compact group, acting faithfully on $\mathfrak{m}$, implies that

(i) $\mathfrak{g}$ is *effective* in the sense that $\mathfrak{h}$ cannot contain a non-trivial ideal $\mathfrak{h}'$ of $\mathfrak{g}$, since this would entail $Ax=[A,x]=0$, whenever $A\in\mathfrak{h}'$ and $x\in\mathfrak{m}$;

(ii) $\mathfrak{g}$ is an *orthogonal* symmetric Lie algebra in the sense that the group of transformations of $\mathfrak{g}$ generated by $\mathrm{ad}(\mathfrak{h})$ is compact, so $\mathfrak{g}$ admits a positive definite $\mathrm{ad}(\mathfrak{h})$-invariant inner product relative to which $\mathfrak{h}$ and $\mathfrak{m}$ are perpendicular.

Now suppose that $H=H(p)$ is the holonomy group of a Riemannian manifold M, with respect to some fixed orthonormal frame p. Then M is said to be *locally symmetric* if its curvature tensor is covariant constant, which means that it is locally constant as a function on the holonomy bundle $Q(p)$ and defines an $\mathfrak{h}$-invariant element $R \in \Re^H$ (cf. **1.3**). Then the above hypotheses are satisfied, where $\mathfrak{m}$ is isomorphic to the tangent space, and the holonomy algebra $\mathfrak{h}$ equals

$$[\mathfrak{m}, \mathfrak{m}] = \mathrm{Im}(R). \tag{5.2}$$

The Killing form of the Lie algebra $\mathfrak{g}$ is the bilinear form

$$K^{\mathfrak{g}}(X, Y) = \mathrm{trace}^{\mathfrak{g}}((\mathrm{ad}\, X)(\mathrm{ad}\, Y)), \quad X, Y \in \mathfrak{g},$$

and (5.1) renders $\mathfrak{h}$ and $\mathfrak{m}$ perpendicular relative to $K^{\mathfrak{g}}$. The restriction of $K^{\mathfrak{g}}$ to $\mathfrak{h}$ is related to the Killing forms of $\mathfrak{h}$ and $\mathfrak{so}(n)$ by

$$\begin{aligned} K^{\mathfrak{g}}(A, B) &= \mathrm{trace}^{\mathfrak{h}}((\mathrm{ad}\, A)(\mathrm{ad}\, B)) + \mathrm{trace}^{\mathfrak{m}}(AB) \\ &= K^{\mathfrak{h}}(A, B) + \tfrac{1}{n-2} K^{\mathfrak{so}(n)}(A, B), \qquad A, B \in \mathfrak{h}. \end{aligned}$$

and is, in particular, negative definite.

The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{m}$ is especially relevant when the manifold M is irreducible, which means that its holonomy group H acts irreducibly on $\mathfrak{m}$. In this case, one can show that $\mathfrak{g}$ is *semisimple*, which means that $K^{\mathfrak{g}}$ is non-degenerate. In fact, any effective symmetric Lie algebra $\mathfrak{h} \oplus \mathfrak{m}$ with $\mathrm{ad}\, \mathfrak{h}$ acting irreducibly on $\mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \neq 0$ is the direct sum of at most two simple ideals [KN, chapter 11, proposition 7.5]. In any case, the restriction of $K^{\mathfrak{g}}$ to $\mathfrak{m}$ is also non-degenerate, and must equal a non-zero constant $1/c$ times the Riemannian metric g induced on $\mathfrak{m}$, since both bilinear forms are H-invariant. Hence

$$\begin{aligned} g(R_{xy}z, w) &= c\, K^{\mathfrak{g}}([[x, y]z], w) \\ &= c\, K^{\mathfrak{g}}([x, y], [z, w]). \end{aligned} \tag{5.3}$$

The Ricci tensor must also be non-zero multiple of g, and regarding R as an element of $\bigodot^2 \mathfrak{h} \cong \bigodot^2 \mathfrak{h}^*$, we obtain the following result, due to Kostant.

5.2 Corollary *The curvature tensor of an irreducible locally symmetric space M has the form*

$$R = c\, K^{\mathfrak{g}} = c(K^{\mathfrak{h}} + \tfrac{1}{n-2} K^{\mathfrak{so}(n)}),$$

and M is Einstein with non-zero scalar curvature.

The Lie algebra $\mathfrak{g}$ is that of the group of isometries of a Riemannian manifold upon which M is modelled, and before describing this in the next section, we first point out an alternative description of $\mathfrak{g}$. On any Riemannian manifold, an *infinitesimal isometry* or *Killing vector field* X is a vector field whose local one-parameter group of diffeomorphisms preserves the metric, which is equivalent to the assertion that relative to any orthonormal frame $p\colon \mathfrak{m} \to T_mM$, the covariant derivative $p^{-1}(\nabla X) \in \mathfrak{m}^*\otimes\mathfrak{m}$ is a skew-symmetric endomorphism.

In general, all one can say is that $p^{-1}(\nabla X)$ lies in the normalizer of the holonomy algebra $\mathfrak{h}$ in $\mathfrak{so}(n)$. However, a calculation involving curvature shows that the component of ∇X orthogonal to $\mathfrak{h}$ determines a *covariant constant* endomorphism of the tangent bundle TM, and this allows one to deduce that the endomorphism ∇X belongs to $\mathfrak{h}$, at least when M is irreducible with non-zero Ricci tensor, or compact. Moreover, in these circumstances, X is completely determined by the pair of values

$$(p^{-1}(\nabla X), p^{-1}X) \in \mathfrak{h}\oplus\mathfrak{m}, \tag{5.4}$$

representing an infinitesimal rotation and translation respectively. The Lie bracket $\bigwedge^2\mathfrak{m}\to\mathfrak{h}$ can then be computed by means of the covariant derivative $p^{-1}(\nabla[X,Y])$ of the Lie bracket of vector fields. More details may be found in the fundamental paper of Kostant [Ko_1].

We have seen that the Lie bracket $\bigwedge^2\mathfrak{m} \to \mathfrak{h}$ of an orthogonal symmetric algebra can be encoded into an invariant "curvature tensor" that belongs to the kernel of the wedging mapping $a\colon \bigodot^2(\Lambda^2) \to \Lambda^4$. If H is a closed subgroup of $SO(n)$ acting irreducibly on $\mathbb{R}^n$, for which the projection t of $\mathfrak{R}^H$ to the subspace $\mathbb{R}$ of **4.3** is non-zero, then the 1-dimensional image of the adjoint t^* is invariant by H. This provides a sort of converse to **5.2**, namely if $\mathfrak{R}^H$ contains an element R with non-zero scalar curvature, then H is the holonomy group of a symmetric space. In fact, case-by-case studies will show us later that if H is not the holonomy group of a symmetric space, then any $R\in\mathfrak{R}^H$ necessarily has zero *Ricci* tensor.

Irrespective of the representation $\mathfrak{h}\hookrightarrow\Lambda^2$, an obvious candidate for an invariant curvature tensor (the only one if H is simple) is the Killing form $K^{\mathfrak{h}} \in \bigodot^2\mathfrak{h}$. We can exploit this situation with the hypothesis that H is the holonomy group of a Riemannian manifold for which $K^{\mathfrak{h}}$ does not lie in $\mathfrak{R}^H$. Then $a(K^{\mathfrak{h}})$ is non-zero and, by association to the holonomy bundle, gives rise to a 4-form Ω on M.

5.3 Lemma *The above procedure defines a nowhere-zero covariant constant* 4*-form* Ω *on* M*, except possibly when* $H\subset O(n)$ *is the isotropy representation of a symmetric space.*

We call Ω the *fundamental 4-form* on M associated to the holonomy reduction. As a corollary, a compact Riemannian manifold M with a holonomy reduction that does not arise from a symmetric space must have its fourth Betti number b_4 non-zero, for otherwise $\Omega = d\alpha$ and $0 = \langle d^* d\alpha, \alpha\rangle = \|\Omega\|^2$. Even if $H \subset O(n)$ is the isotropy representation of a symmetric space, there is a good chance that Ω will be non-zero, at least if H is not simple. On a Kähler manifold with $H = U(m)$, **5.2** may be used to show that the 4-form Ω is a constant multiple of the square $\omega^2 = \omega \wedge \omega$ of the symplectic 2-form. In the sequel, we shall meet different types of 4-forms arising from various holonomy groups.

Homogeneous Spaces

Consider the problem of constructing a Riemannian manifold from a Lie algebra of the form

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \qquad [\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}. \tag{5.5}$$

Let $\widetilde{G}$ be the connected, simply-connected Lie group with Lie algebra $\mathfrak{g}$, and let $\widetilde{H}$ be the connected Lie subgroup with subalgebra $\mathfrak{h}$. At this point, we have generalized the previous assumptions by allowing the $\mathfrak{m}$-component of $[\mathfrak{m}, \mathfrak{m}]$ to be non-zero, and its $\mathfrak{h}$-component to equal a proper subset of $\mathfrak{h}$. However we do impose the hypotheses:

(i) $\mathfrak{h}$ contains no non-trivial ideal of $\mathfrak{g}$;

(ii) $\widetilde{H}$ is compact, and a closed subgroup of $\widetilde{G}$.

The space of left cosets $M = \widetilde{G}/\widetilde{H}$ is a simply-connected manifold. There is no reason to suppose that the natural left action of $\widetilde{G}$ on M is effective, but (i) implies that its kernel $\widetilde{Z} = \{g \in \widetilde{G} : gm = m, \forall m \in M\}$ is a discrete central subgroup contained in $\widetilde{H}$, and

$$M = G/H, \quad \text{with} \quad G = \widetilde{G}/\widetilde{Z}, \quad H = \widetilde{H}/\widetilde{Z}.$$

Then $\mathfrak{m}$ is identified with the tangent space to M at the identity coset $o = eH$ via the composition

$$\mathfrak{m} \hookrightarrow \mathfrak{g} \cong T_e G \xrightarrow{\pi_*} T_o M,$$

where $\pi\colon G \to M$ is the projection. The adjoint action of H on $\mathfrak{m}$ determines the *isotropy representation*, that measures the derivative of the action of H at o. By (i), the ideal $\{A \in \mathfrak{h} : [A, x] = 0, \forall x \in \mathfrak{m}\}$ of $\mathfrak{g}$ is zero, and using the exponential mapping in G, one can show that the isotropy representation is faithful. By (ii), $\mathfrak{m}$ admits an H-invariant inner product, and the choice of an orthonormal basis of $\mathfrak{m}$ determines a frame $p \in LM$. The induced action of G on LM describes a principal H-subbundle containing p isomorphic to G itself; as $H \subseteq O(n)$, this is contained in a unique $O(n)$-bundle P consisting of the orthonormal frames defining a G-invariant Riemannian metric on M.

The left translates $\{L_{g*}\mathfrak{m} : g \in G\}$ of the subpace $\mathfrak{m}$ of T_eG constitute a horizontal distribution on the principal bundle $G \subset P$. Not only is this distribution left-invariant by G, but it is also *right*-invariant by H, and therefore determines a connection on LM, the so-called *canonical connection of the second kind* [N_1]. This fact is a consequence of our assumption that the homogeneous space M is *reductive* in the sense that $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$, for this implies

$$R_{h*}(L_{g*}\mathfrak{m}) = L_{gh*}(\mathrm{Ad}\, h^{-1})\mathfrak{m}, \quad g \in G,\ h \in H.$$

The sum $\phi + \theta$ of the 1-forms on LM defined in chapter 1 now coincides with the left-invariant Maurer-Cartan form on G. Consequently, the curvature and torsion (1.13) of the above connection can be identified with the linear maps

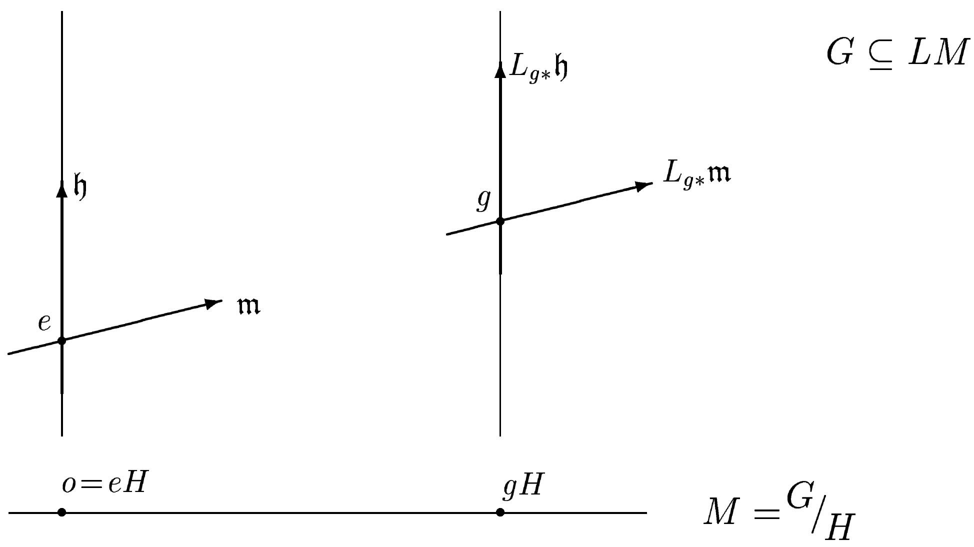
$$\begin{aligned} &\textstyle\bigwedge^2\mathfrak{m} \longrightarrow \mathfrak{h}, \\ &\textstyle\bigwedge^2\mathfrak{m} \longrightarrow \mathfrak{m}, \end{aligned}$$

determined by the respective components of $[\mathfrak{m}, \mathfrak{m}]$.

In terms of (1.20), the $\mathfrak{m}$-component of $[\mathfrak{m}, \mathfrak{m}]$ may be interpreted as an invariant element ξ of $\mathfrak{m}^* \otimes \mathfrak{gl}(n, \mathbb{R})$, and vanishes if and only if the Lie algebra (5.5) is symmetric. On the other hand, the modified form $\phi + \theta\xi$ always determines a torsion-free connection on G/H, called the *canonical connection of the first kind*. The two canonical connections share the same geodesics, which are the projections of one-parameter subgroups in G, and are therefore complete. The canonical connection of the first kind coincides with the Levi Civita connection if and only if $\xi \in \mathfrak{m}^* \otimes \mathfrak{so}(n)$, which is equivalent to the condition

$$g(x, [y, z]_{\mathfrak{m}}) = g([x, y]_{\mathfrak{m}}, z), \quad x, y, z \in \mathfrak{m}. \tag{5.6}$$

5.4 Figure Canonical connection of the second kind



A Riemannian homogeneous space G/H together with a splitting (5.5) satisfying (5.6) is called *naturally reductive*, and its Riemann curvature tensor is not a great deal more complicated than **5.2**. An important subclass consists of the *normal spaces* for which the splitting (5.5) is orthogonal relative to an $\operatorname{Ad} G$-invariant inner product on $\mathfrak{g}$, for example the Killing form $K^{\mathfrak{g}}$ if G is compact. These spaces provide a rich source of Einstein metrics, many of which have been classified by Wang and Ziller [WZ],[Bes_2, chapter 7] and generalize the metrics on symmetric and isotropy irreducible spaces.

When (5.5) is an orthogonal symmetric Lie algebra, the holonomy group of G/H is not necessarily equal to H; rather it is the subgroup of H with Lie algebra (5.2). The holonomy does equal H if G is semisimple, for in this case the orthogonal complement with respect to $K^{\mathfrak{g}}$ of the ideal $[\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{m}$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$, and must vanish given our assumption that G acts effectively. Combining results above with a de Rham type decomposition theorem leads to the following characterization.

5.5 Theorem *A complete simply-connected Riemannian manifold with $\nabla R = 0$ admits a transitive connected group G of isometries, whose isotropy subgroup H coincides with the holonomy group. Moreover, H is the connected component of the fixed point set of an involutive automorphism σ of G.*

A *Riemannian globally symmetric space* is a coset manifold G/H, where G is a connected Lie group with an involutive automorphism σ, and H is a closed subgroup lying between the fixed point set of σ and its identity component, with $\mathrm{Ad}\,H$ a compact group of transformations of $\mathfrak{g}$. For each coset $m = gH$, the mapping

$$g'H \mapsto g\sigma(g^{-1}g')H$$

determines an isometry of G/H, and the fact that its square is the identity can be used to show that m is an isolated fixed point. This isometry coincides with the *geodesic symmetry* s_m defined by changing the sign of the parameter of geodesics emanating from m. Such an s_m is defined in a neighbourhood of any point m of any Riemannian manifold, and satisfies $s_m^* \nabla R = -\nabla R$, because ∇R is a tensor of odd order. The locally symmetric condition $\nabla R = 0$ is characterized by s_m being an isometry, which extends globally only when M has the form G/H above [BL$_2$].

The simplest, but in some sense universal, instance of the above constructions starts from the space $\mathfrak{R} = \mathfrak{R}^{O(n)}$ of curvature tensors of a generic Riemannian manifold. This space contains an invariant element described by **4.3** and (4.7), which is essentially the induced inner product on the space $\bigwedge^2\mathbb{R}^n$. The corresponding symmetric Lie algebra is

$$\begin{aligned} \mathfrak{so}(n) \oplus \mathbb{R}^n &\cong \textstyle\bigwedge^2(\mathbb{R}^n \oplus \mathbb{R}) \\ &\cong \mathfrak{so}(n+1), \end{aligned} \tag{5.7}$$

and the coset space M is the n-dimensional sphere

$$S^n = \frac{Spin(n+1)}{Spin(n)} = \frac{SO(n+1)}{SO(n)}.$$

Observe that as the scalar curvature tends to zero, the left-hand side of (5.7) becomes the Lie algebra of the symmetric space $\mathbb{R}^n = E(n)/O(n)$, where $E(n)$ is the group of Euclidean motions, although this description is not in line with **5.5**. In the case of the sphere, it is easy to find a *linear* representation of the isometry group G (namely the basic one of $SO(n+1)$ on $\mathbb{R}^{n+1}$) whose isotropy representation coincides with that of M. For a general space $M = G/H$, such a linear representation of G always exists, and gives an embedding of M as an orbit in a higher dimensional sphere [Mos].

Inclusions between classical groups

The group $U(m)$ consists of complex linear transformations preserving a Hermitian form. Similarly, the quaternionic unitary group $Sp(k)$ is the set of transformations of $\mathbb{H}^k$ commuting with right multiplication by quaternion scalars, and preserving the quaternionic form $\sum_{\alpha=1}^k \overline{dq}^\alpha \otimes dq^\alpha$. This may be analyzed by writing

$$dq^\alpha \;=\; dz^\alpha + j dw^\alpha \;=\; dx_1^\alpha + i dx_2^\alpha + j dx_3^\alpha + k dx_4^\alpha, \quad \alpha = 1, \dots, k$$

successively in complex and real coordinates.

In terms of the underlying complex vector space $\mathbb{C}^{2k}$ or its dual $(\mathbb{C}^{2k})^* \cong \lambda^{1,0}$, the group $Sp(k)$ is the subgroup of $U(2k)$ of elements commuting with the antilinear map ε induced by right multiplication by j on $\mathbb{H}^k$, so that in coordinates,

$$\varepsilon\colon (z^\alpha, w^\alpha) \mapsto (-\overline{w}^\alpha, \overline{z}^\alpha).$$

If A, B are complex $k \times k$-matrices, then in analogy to (3.2), the mapping

$$A + jB \mapsto \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix},$$

determines the left-hand inclusion

$$\begin{array}{ccccc} & & U(2k) & \hookrightarrow & SO(4k) \\ & & \cup & & \\ Sp(k) & \hookrightarrow & SU(2k) & & \end{array} \tag{5.8}$$

$$\text{cf.} \qquad \mathbb{H}^k \qquad = \qquad \mathbb{C}^{2k} \qquad = \qquad \mathbb{R}^{4k}$$

of $Sp(k)$ into the special unitary group, whose elements have determinant one.

The operator ε may be thought of as a *linear* map $\lambda^{1,0} \to \lambda^{0,1}$ that makes the two representations $\lambda^{1,0}$, $\lambda^{0,1}$, albeit distinct for $U(2k)$, isomorphic relative to the subgroup $Sp(k)$. In an attempt to continue our previous conventions, we shall denote the common complex $Sp(k)$-representation

$$\lambda^{1,0} \cong \lambda^{0,1} \cong (\lambda^{1,0})^*$$

by λ^1, and its exterior and symmetric powers by $\lambda^k = \bigwedge^k(\lambda^1)$ and $\sigma^k = \bigodot^k(\lambda^1)$. For example,

$$\begin{aligned} \operatorname{End}\mathbb{C}^{2k} \cong \lambda^{1,0}\otimes\lambda^{0,1} &= \mathbb{C}\oplus\lambda_0^{1,1} \\ &= \mathbb{C}\oplus\lambda_0^2\oplus\sigma^2, \end{aligned} \tag{5.9}$$

where the 1-dimensional summand in the last line is spanned by the complex symplectic form $2\sum_{\alpha=1}^k dz^\alpha\wedge dw^\alpha$, equal to the j component of $\sum_{\alpha=1}^k \overline{dq}^\alpha\otimes dq^\alpha$.

The two preceding paragraphs present different ways of understanding the action of ε; first it was treated like an almost complex structure, and then its antilinear character made it out to be a type of complex conjugation, but acting on λ^1 with square -1 rather than $+1$. Indeed, ε (or to be more accurate $\varepsilon\otimes\varepsilon$) is an honest complex conjugation on $\lambda^1\otimes\lambda^1$, because $(\varepsilon\otimes\varepsilon)^2 = +1$. More generally, a representation of a compact group G on a complex m-dimensional vector space V is called *real* (respectively *quaternionic*) if G commutes with an antilinear map ε on V with square $+1$ (respectively -1). If we choose a Hermitian form on V as in (3.1), and average it by integration over G and $\mathbb{Z}_2 = \{1,\varepsilon\}$, the result will be a real G-invariant non-degenerate element of $\bigwedge^2 V^*$ (respectively $\bigodot^2 V^*$). In other words, the representation will not only be unitary, but it will factor through the group $O(m)$ (respectively $Sp(\frac{m}{2})$).

The fixed points of a real structure ε on V define a real vector space which we denote by $[V]$, whose complexification is V. For example, every summand of (5.9) is the complexification of a real vector space, and in analogy with (2.2),

$$\mathfrak{sp}(k)\otimes_{\mathbb{R}}\mathbb{C}\cong\sigma^2, \qquad \text{or} \qquad \mathfrak{sp}(k)\cong[\sigma^2]. \tag{5.10}$$

In general, in the absence of any ε, the only reliable way of manufacturing a real representation is to ignore the action of the scalar i, so as to obtain the *underlying* real vector space. We denote this by $[\![V]\!]$; its real dimension is twice the complex dimension of V, and

$$[\![V]\!]\otimes_{\mathbb{R}}\mathbb{C}\cong V\oplus\overline{V}, \quad \text{or} \quad [\![V]\!]\cong[V\oplus\overline{V}].$$

The definitions are consistent with (3.4).

The antilinear maps with square ±1 place the left-hand inclusions in (5.8) and

$$\begin{array}{ccccc} & & U(m) & \hookrightarrow & Sp(m) \\ & & \cup & & \\ SO(m) & \hookrightarrow & SU(m) & & \end{array} \tag{5.11}$$

$$\text{cf.} \qquad \mathbb{R}^m \quad \subset \quad \mathbb{C}^m \quad \subset \quad \mathbb{H}^m$$

on the same footing. All four horizontal inclusions furnish symmetric Lie algebras, with corresponding symmetric spaces

$$\frac{SO(2m)}{U(m)}, \quad \frac{Sp(m)}{U(m)}, \quad \frac{SU(2k)}{Sp(k)}, \quad \frac{SU(m)}{SO(m)}. \tag{5.12}$$

The first space parametrizes almost complex structures on $\mathbb{R}^{2m}$ that are orthogonal and compatible with a fixed orientation; its symmetric Lie algebra is the decomposition (3.5) that played such a crucial role in the study of $U(m)$-structures, and whose isotropy representation $[\![\lambda^{2,0}]\!]$ featured with good reason in **3.3**. The remaining three spaces correspond to the Lie algebras

$$\begin{aligned} \mathfrak{sp}(m) &\cong \mathfrak{u}(m) \oplus [\![\sigma^{2,0}]\!], \\ \mathfrak{su}(2k) &\cong \mathfrak{sp}(k) \oplus [\lambda_0^2], \\ \mathfrak{su}(m) &\cong \mathfrak{so}(m) \oplus \Sigma_0^2, \end{aligned} \tag{5.13}$$

where $\sigma^{2,0} = \bigodot^2(\lambda^{1,0})$, and Σ_0^2 is defined by (2.2).

Further examples and properties

The fact that distinct symmetric spaces may share the same isometry group (witness the last two families in (5.12)) is clarified by the notion of duality. Any symmetric Lie algebra $\mathfrak{g}$ has a *dual* $\mathfrak{g}^*$ defined by reversing the sign of the curvature operator $\bigwedge^2\mathfrak{m} \to \mathfrak{h}$, so that as a subalgebra of the complexification of $\mathfrak{g}$,

$$\mathfrak{g}^* = \mathfrak{h} \oplus i\mathfrak{m}.$$

The resulting involution on $\mathfrak{g}^*$ is the restriction of the *complex conjugation* whose fixed points define the real form $\mathfrak{g}$. Clearly, the symmetric Lie algebra $\mathfrak{g}$ is itself the dual of $\mathfrak{g}^*$.

Applying the above procedure to the algebras underlying (5.12) yields the dual symmetric spaces

$$\frac{SO(m,\mathbb{H})}{U(m)},\quad \frac{Sp(m,\mathbb{R})}{U(m)},\quad \frac{SL(k,\mathbb{H})}{Sp(k)},\quad \frac{SL(m,\mathbb{R})}{SO(m)}. \tag{5.14}$$

Here $SO(m,\mathbb{H})$ is the intersection $SL(m,\mathbb{H})\cap SO(2n,\mathbb{C})$ inside $GL(2m,\mathbb{C})$, and its coset space may be interpreted as the set of all complex structures on $\mathbb{R}^{2m}$ for which the corresponding totally isotropic subspace V of $\mathbb{C}^{2m}\cong\mathbb{H}^m$ satisfies $V\cap Vj=\{0\}$. The double-covering $SL(4,\mathbb{C})\to SO(6,\mathbb{C})$ (cf. **6.2**) identifies $SL(2,\mathbb{H})/Sp(2)\cong SO_0(5,1)/SO(5)$ with the real hyperbolic 5-space. The latter may be thought of as the group of quaternionic projective or conformal transformations modulo the subgroup of isometries, and as such is the space of basic self-dual Yang-Mills fields on S^4 modulo gauge equivalence [AHS].

A first step in an investigation of the structure of an arbitrary orthogonal symmetric Lie algebra needed to prove **5.5** is to show, in analogy to **2.9**, that it may be decomposed as a σ-invariant direct sum of a "Euclidean" algebra for which $[\mathfrak{m},\mathfrak{m}]=0$, and "irreducible algebras" for which $\mathfrak{g}$ is semisimple and contains $\mathfrak{h}$ as a maximal proper subalgebra. Every irreducible algebra is either of of *compact* or *non-compact type*, meaning that the restriction of the Killing form $K^{\mathfrak{g}}$ to $\mathfrak{m}$ is negative definite ($c<0$ in (5.3)) or positive definite ($c>0$). In the former case, $K^{\mathfrak{g}}$ is negative definite on the whole of $\mathfrak{g}$ which is equivalent to the compactness of the corresponding Lie group, and either $\mathfrak{g}$ is simple ("type I"), or the sum of two simple ideals interchanged by σ ("type II"). The irreducible algebras of non-compact type ("III" and "IV") may then be obtained via duality, although are in some ways easier to describe directly. We refer the reader to [KN, chapter 11] and [He] for details of the key facts that we are summarizing.

Using the theory of roots, one can prove that any simple complex Lie algebra is the complexification $\mathfrak{g}_{\mathbb{C}}$ of a Lie algebra $\mathfrak{g}$ of some compact group [Wey]. This *compact real form* $\mathfrak{g}$ is unique up to an inner automorphism of $\mathfrak{g}_{\mathbb{C}}$, and the problem of finding other real forms of $\mathfrak{g}_{\mathbb{C}}$ becomes equivalent to finding involutive automorphisms of $\mathfrak{g}$. Indeed, given another real form $\mathfrak{g}^*$, one can choose the compact form $\mathfrak{g}$ so that

$$\begin{aligned}\mathfrak{g}^* &= (\mathfrak{g}^*\cap\mathfrak{g})\oplus(\mathfrak{g}^*\cap i\mathfrak{g}),\\ \mathfrak{g} &= (\mathfrak{g}^*\cap\mathfrak{g})\oplus(i\mathfrak{g}^*\cap\mathfrak{g}),\end{aligned} \tag{5.15}$$

is a pair of orthogonal symmetric Lie algebras of non-compact and compact type respectively. The first of these defines the so-called *Cartan decomposition* of $\mathfrak{g}^*$ [C$_5$], and exhibits $\mathfrak{g}^* \cap \mathfrak{g}$ as a maximal Lie subalgebra of $\mathfrak{g}^*$ on which $K^{\mathfrak{g}^*}$ is negative definite. In fact, a simple Lie group G^* has a maximal compact subgroup H, unique up to conjugacy [C_4], and G^*/H is diffeomorphic to Euclidean space. The conjugacy is equivalent to the assertion that any compact subgroup of G^* has a fixed point on G^*/H, and may be proved by exploiting the negative curvature.

Let H be a simply-connected compact simple Lie group. Then $G = H \times H$ acts on H by $(x, y)h = xhy^{-1}$, and the stabilizer at the identity is the diagonal subgroup. The Lie bracket makes

$$\begin{aligned} \mathfrak{g} &= \{(A, A) : A \in \mathfrak{h}\} \oplus \{(A, -A)\} &\cong \mathfrak{h} \oplus \mathfrak{h}, \\ \mathfrak{g}^* &= \{(A, A)\} \oplus \{(iA, -iA)\} &\cong \mathfrak{h} \oplus i\mathfrak{h} \end{aligned} \tag{5.16}$$

into symmetric Lie algebras, the second of which is the real algebra underlying the complexification of $\mathfrak{h}$. The corresponding symmetric spaces are H itself, endowed with a bi-invariant Riemannian metric, and $H_{\mathbb{C}}/H$, where $H_{\mathbb{C}}$ is the corresponding simple complex Lie group. Once again, the space of non-compact type arises from a maximal compactly embedded subalgebra of a real simple Lie algebra $\mathfrak{g}^*$, but this time $\mathfrak{g}^*$ admits an almost complex structure so that its complexification is no longer simple.

5.6 Theorem [C_5] *The recipes* (5.15), (5.16) *establish a bijective correspondence between real Lie algebras, whose complexification is simple, and pairs of simply-connected irreducible Riemannian symmetric spaces.*

From the point of view of holonomy, it follows that any semisimple compact connected centreless Lie group H arises, via its adjoint representation

$$H \hookrightarrow SO(\dim H), \tag{5.17}$$

as the holonomy group of a Riemannian manifold. Incidentally, when H is simple, $SO(\dim H)/H$ has irreducible isotropy representation, and in general admits a normal homogeneous Einstein metric [W_4],[WZ]. In contrast to the symmetric description of H, note that the splitting

$$\mathfrak{g} = \{(A, A) : A \in \mathfrak{h}\} \oplus \{(0, A)\}$$

makes H into a reductive homogeneous space, for which the connection **5.4** has zero curvature, and torsion given by Lie bracket.

Although the last two families of (5.12) both arise from real forms of the complex special linear group, it is in many ways the first two that have more in common. The spaces in question both admit a complex structure which originates from the non-discrete centre of their common holonomy group $U(m)$. The same is true of complex projective space $\mathbb{C}P^m$ (3.14), for which $U(m)$ acts in the standard way on the tangent space. Since $\mathfrak{u}(m+1) \cong [(\lambda^{1,0} \oplus \mathbb{C})^* \otimes (\lambda^{1,0} \oplus \mathbb{C})]$, the symmetric Lie algebra for $\mathbb{C}P^m$ can be expressed in terms of $U(m)$-modules as

$$\begin{aligned} \mathfrak{su}(m+1) &\cong [\lambda^{1,1} \oplus \lambda^{1,0} \oplus \lambda^{0,1}] \\ &\cong \mathfrak{u}(m) \oplus [\![\lambda^{1,0}]\!]. \end{aligned}$$

A symmetric space $M = G/H$ corresponding to an orthogonal Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is called *Hermitian* if $\mathfrak{m}$ possesses an H-invariant almost complex structure I, which acts as an orthogonal transformation with respect to the metric. In these circumstances, I extends to a G-invariant almost complex structure on M which is parallel with respect to the Levi Civita connection, and therefore makes M into a Kähler manifold.

If $\mathfrak{h}'$ denotes the subalgebra of $\mathfrak{so}(n)$ generated by $\mathfrak{h}$ and I, then $\mathfrak{h}' \oplus \mathfrak{m}$ is also a symmetric Lie algebra. If G is semisimple, an argument resembling that immediately preceding 5.5 shows that $\mathfrak{h}' = \mathfrak{h}$, whence I belongs to $\mathfrak{h}$, and therefore to the *centre* of $\mathfrak{h}$. Conversely, provided H acts faithfully on $\mathfrak{m}$, then any element of order 4 in its centre will define an invariant almost complex structure I. The compact irreducible Hermitian symmetric spaces are then the spaces G/H, where G is a compact simple connected centreless Lie group, and H is a maximal connected proper subgroup of G with non-discrete centre [BL$_2$].

The complexified Lie algebra defining a Hermitian symmetric space has the form

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{m}^{1,0} \oplus \mathfrak{m}^{0,1}, \tag{5.18}$$

where $\mathfrak{m}^{0,1}$, $\mathfrak{m}^{0,1}$ are the isotropic i, $-i$ eigenspaces of I. Because $[\mathfrak{m},\mathfrak{m}] \subseteq \mathfrak{h}$, the component of $[\mathfrak{m}^{0,1},\mathfrak{m}^{0,1}]$ in $\mathfrak{m}^{1,0}$ is certainly zero; geometrically, this corresponds to the fact that the Lie bracket of any two vector field of type $(0,1)$ is again of type $(0,1)$. The integrability of the complex structure can be understood directly by observing that

$$\mathfrak{p} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{m}^{0,1} \tag{5.19}$$

is a complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let $G_{\mathbb{C}}$ be the simply-connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, and let G, H, P denote the connected subgroups of $G_{\mathbb{C}}$ arising from g, $\mathfrak{h}$, $\mathfrak{p}$, regarded as subalgebras of the real Lie algebra underlying $\mathfrak{g}_{\mathbb{C}}$. The Borel embedding theorem [B$_3$] asserts that the natural mapping

$$gH \mapsto gP, \quad g \in G$$

exhibits G/H as an open complex submanifold of the complex coset space $G_{\mathbb{C}}/P$. A Hermitian symmetric space $G/H = G_{\mathbb{C}}/P$ of compact type contains its dual G^*/H as an open G^*-orbit; complex hyperbolic m-space $SU(m,1)/S(U(m)\times U(1))$ is identified with an affine subset of $\mathbb{C}P^m$. More generally, each Hermitian symmetric space of non-compact type can be identified with a *bounded symmetric domain*, that is a bounded open connected subset D of $\mathbb{C}^m$, for which each $m \in D$ is the isolated fixed point of a holomorphic diffeomorphism $s\colon D \to D$, whose square equals the identity [C$_6$].

The actual classification of irreducible symmetric spaces G/H is given in [C$_4$]; the table lists the simply-connected ones of compact type, partly in terms of second homotopy groups which may be computed from the homotopy exact sequence

$$\pi_2(H) \longrightarrow \pi_2(G) \longrightarrow \pi_2(M) \longrightarrow \pi_1(H) \longrightarrow \pi_1(G) \longrightarrow 0,$$

implications of which appear in [BR]. There may be several symmetric spaces with the same compact universal covering G/H, depending upon the size of the centre of G. There is only one when G is one of G_2, F_4, E_8, or when G/H is Hermitian symmetric. In order to achieve visual simplicity, some spaces below are presented with an isometry group which is neither simply-connected nor effective. The feature common to the ones labelled "quaternionic" is that the isotropy subgroup contains a factor isomorphic to $Sp(1)$ or $SO(3)$ in the same way that the Hermitian spaces have a $U(1)$ or $SO(2)$ in their isotropy; their theory will be pursued in chapter 9.

5.7 Table Compact simply-connected irreducible Riemannian symmetric spaces

Lie group $\pi_2(M)=0$	Hermitian $\pi_2(M)=\mathbb{Z}$	Quaternionic $\pi_2(M)=\mathbb{Z}_2$ except $*$	Others $\pi_2(M)=0$ or $\mathbb{Z}_2$
$SU(m)$	$\dfrac{SU(k+m)}{S(U(k)\times U(m))}$	$\dfrac{{}^*SU(k+2)}{S(U(k)\times U(2))}$	$\dfrac{SU(m)}{SO(m)}, \dfrac{SU(2k)}{Sp(k)}$
$Spin(n)$	$\dfrac{SO(2m)}{U(m)}, \dfrac{SO(m+2)}{SO(m)\times SO(2)}$	$\dfrac{SO(k+4)}{SO(k)\times SO(4)}$	$\dfrac{SO(k+m)}{SO(k)\times SO(m)}$
$Sp(k)$	$\dfrac{Sp(k)}{U(k)}$	$\dfrac{{}^*Sp(k+1)}{Sp(k)\times Sp(1)}$	$\dfrac{Sp(k+m)}{Sp(k)\times Sp(m)}$
E_6	$\dfrac{E_6}{Spin(10)U(1)}$	$\dfrac{E_6}{SU(6)Sp(1)}$	$\dfrac{E_6}{F_4}, \dfrac{E_6}{Sp(4)_\sim}$
E_7	$\dfrac{E_7}{E_6U(1)}$	$\dfrac{E_7}{Spin(12)Sp(1)}$	$\dfrac{E_7}{SU(8)_\sim}$
E_8		$\dfrac{E_8}{E_7Sp(1)}$	$\dfrac{E_8}{Spin(16)_\sim}$
F_4		$\dfrac{F_4}{Sp(3)Sp(1)}$	$\dfrac{F_4}{Spin(9)}$
G_2		$\dfrac{G_2}{SO(4)}$	

Juxtaposition AB of two groups generally denotes the quotient $A\times_{\mathbb{Z}_2}B$

$\sim$ tagged on an isotropy group denotes its quotient $/\mathbb{Z}_2$

6 Representation Theory

One aim of this chapter is to explain how to decompose the tensor product of two irreducible representations of a compact Lie group. A standard method for doing this lends itself well to computer treatment, but there is much to be gained by learning to perform the computations by hand without a great deal of subtlety and, in any case, many geometrically significant representations are relatively simple. The relevant theory is well documented, so our approach concentrates on the practicalities needed to give the reader quick access to results in the sequel.

To begin with, we summarize the pertinent facts concerning representations of compact Lie groups, stating many results without proofs, which may be consulted in, for example, [Ad] or [BD]. At opportune moments, there is reference to the link between orbits in the adjoint representation and complex homogeneous spaces. More detail is injected into the section on examples, which bases an illustration of the theory on the low-dimensional spin groups, and their mutual inclusion in the orthogonal group $SO(8)$, a theme which will be taken up again towards the end of the notes.

Weights and roots

Throughout this chapter, G denotes a compact connected Lie group, and a representation of G signifies a continuous, or equivalently smooth, group homomorphism $\rho\colon G \to \mathrm{Aut}_{\mathbb{C}} V$, where V is a *complex* vector space. Taking V to be complex is no great restriction, as a real or quaternionic structure may be imposed in the manner explained after (5.9). The derivative of ρ defines the corresponding Lie algebra representation

$$d\rho\colon T_eG \cong \mathfrak{g} \longrightarrow \mathrm{End}\, V,$$

which interprets the Lie bracket on $\mathfrak{g}$ as the usual commutator on $\mathrm{End}\, V$. A fundamental example is the *adjoint representation* $\rho = \mathrm{Ad}$ of G on $V = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, which is derived from conjugation, and satisfies $\exp(\rho(h)X) = h \exp(X) h^{-1}$. In this case $d\rho(X)Y$ equals $(\mathrm{ad}\, X)Y = [X, Y]$ is the operation of Lie bracket, its representation-theoretic character corresponding to the Jacobi identity (1.1).

A *torus* is a connected closed abelian subgroup of G. We fix a maximal one T, not properly contained in any other torus. Any two maximal tori are conjugate, in analogy to the result concerning maximal compact subgroups of a semisimple group, mentioned in connection with symmetric spaces. The Lie algebra $\mathfrak{t}$ of T is a maximal abelian Lie subalgebra of $\mathfrak{g}$, and its dimension defines the *rank* of G. In fact, the intersection of $\mathfrak{t}$ with any orbit of the adjoint representation of G on $\mathfrak{g}$ is a finite set of points, itself an orbit of the *Weyl group* $N(T)/T$. Moreover

$$\mathfrak{t} = \{X \in \mathfrak{g} : [X, X_0] = 0\}, \tag{6.1}$$

for a vector X_0 that lies in $\mathfrak{t}$ but no other maximal abelian subalgebra of $\mathfrak{g}$.

Given a representation V of G, the endomorphisms $d\rho(x)$ are simultaneously diagonalizable as x ranges over $\mathfrak{t}$. Hence there is a decomposition

$$V = \bigoplus_{\alpha} V_\alpha, \tag{6.2}$$

where each V_α is a complex $\mathfrak{t}$-invariant subspace, whose label α denotes a real linear form on $\mathfrak{t}$ such that

$$x(v) = 2\pi i\alpha(x), \quad x \in \mathfrak{t},\ v \in V_\alpha.$$

Exponentiation defines a homomorphism from $(\mathfrak{t}, +)$ to T, whose kernel is a lattice of $\mathfrak{t}$, and each α in (6.2) belongs to the dual lattice in the sense that $\alpha(x) \in \mathbb{Z}$ whenever $\exp(x) = e$. An element of $\mathfrak{t}^*$ with this last property is called a *weight*, and the dimension of V_α represents the *multiplicity* of α as a weight of V.

For the adjoint representation $V = \mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}$, it is well known that the only weight which occurs with a multiplicity greater than one is 0, and almost by definition the zero weight space is $\mathfrak{g}_0 = \mathfrak{t}\otimes_{\mathbb{R}}\mathbb{C}$. The non-zero weights that occur constitute the set R of *roots* of $\mathfrak{g}$, and their weight spaces satisfy

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \begin{cases} \mathfrak{g}_{\alpha+\beta}, & \alpha+\beta \in R \\ 0, & \alpha+\beta \notin R \end{cases}. \tag{6.3}$$

There exists a subset Δ of R, comprising the so-called *simple roots*, such that any $\alpha \in R$ can be written $\alpha = \sum_{\beta_i \in \Delta} n_i\beta_i$, where the n_i are integers, all of the same sign. Taking the n_i all positive defines the R^+ of *positive roots*, and

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in R^+} [\![\mathfrak{g}_\alpha]\!]. \tag{6.4}$$

If $(\ ,\ )$ is any inner product on $\mathfrak{t}$ invariant by the action of $N(T)$, the expression

$$\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)} \tag{6.5}$$

is an integer whenever $\alpha \in \mathfrak{t}^*$ is a weight, and β is a root. Then the Weyl group turns out to equal the group generated by reflections

$$\sigma_\beta \colon \alpha \mapsto \alpha - \langle \alpha, \beta \rangle \beta, \quad \alpha \in R,$$

in the hyperplanes of $\mathfrak{t}^*$ orthogonal to the roots, and one can associate to each element σ of the Weyl group its sign $(-1)^\sigma$. A choice of Δ determines the *fundamental (dual) Weyl chamber*

$$\Upsilon = \{\gamma \in \mathfrak{t}^* : (\gamma, \beta) > 0,\ \forall \beta \in \Delta\},$$

Any orbit of G on $\mathfrak{g}$ now has a unique representative in the closure $\overline{\Upsilon}$, and a *weight* γ lying in $\overline{\Upsilon}$ is called *dominant.* An important quantity

$$d = \tfrac{1}{2} \sum_{\alpha \in R^+} \alpha$$

has the enviable property that $\gamma \in \overline{\Upsilon}$ if and only if $\gamma + d \in \Upsilon$. If G is not simply-connected, d may not itself be a weight.

At this point, we mention the homogeneous spaces that arise as orbits of the adjoint representation. In analogy to (5.5), the Lie algebra (6.4) corresponds to the *flag manifold* G/T, whose tangent space is identified with $\mathfrak{m} = \bigoplus_{\alpha \in R^+} [\![\mathfrak{g}_\alpha]\!]$. The choice of an orientation on each 2-dimensional summand $[\![\mathfrak{g}_\alpha]\!]$ gives rise to a G-invariant almost complex structure on G/T, which may or may not be integrable. Such a choice is determined by a set Δ of simple roots, by taking the space of $(1,0)$-vectors equal to

$$\mathfrak{m}^{1,0} = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha;$$

this is closed under Lie bracket and defines a *complex* structure on G/T.

An element $X_0 \in \mathfrak{g}$ is called *regular* if it does not belong to a root hyperplane; in this case its G-orbit is of maximal dimension and has the form G/T (hence (6.1)). Any other orbit is isomorphic to $G/C(\gamma)$ for some dominant weight γ on a wall of

Υ, where $C(\gamma)$ denotes the centralizer of the one-parameter subgroup generated by γ. This leads to the generalization

$$\mathfrak{g} = \Big(\mathfrak{t} \oplus \bigoplus_{(\alpha,\gamma)=0} [\![\mathfrak{g}_\alpha]\!]\Big) \oplus \bigoplus_{(\alpha,\gamma)>0} [\![\mathfrak{g}_\alpha]\!], \tag{6.6}$$

of (6.4). A theorem of Borel [Se] asserts that any $2m$-dimensional compact homogeneous space M which is symplectic (or in particular Kähler) is necessarily of the form $G/C(\gamma)$ for some γ. The embedding $\mu\colon M \hookrightarrow \mathfrak{g} \cong \mathfrak{g}^*$ is the *moment map* constructed from the symplectic structure of M (see chapter 8).

A choice Δ of simple roots again determines a complex structure on $G/C(\gamma)$, which may be described as a complex quotient $G_{\mathbb{C}}/P$, where P is the *parabolic subalgebra* generated by $\mathfrak{t}_{\mathbb{C}}$ and the root spaces $\mathfrak{g}_\alpha$ with $(\alpha,\gamma) \geqslant 0$. A special case is the *Borel subgroup* B generated by $\mathfrak{t}_{\mathbb{C}}$ and *all* the positive roots, and there is a holomorphic fibration

$$G/T \cong G_{\mathbb{C}}/B \longrightarrow G_{\mathbb{C}}/P \cong G/C(\gamma),$$

whose fibre is tangent to the direct sum of terms with $(\alpha,\gamma)=0$ in (6.6). The case of a Hermitian symmetric space, discussed in (5.18) arises when the centre of $C(\gamma)$ is generated by γ. Relevant details are contained in the papers of Wang [Wa] and Bott [Bot].

Irreducible representations

A complex representation V is said to be *irreducible* if it has no proper G-invariant complex subspace. A little caution is needed if V has a real or quaternionic structure, since there may be invariant subspaces which do not inherit the respective structure, so there are weaker notions of *real* or *quaternionic irreducible.* For example, suppose that V is some complex representation, and let $\overline{V}$ denote the *conjugate space* (so its elements are the same as those of V, but scalars act conjugated). Then

$$\varepsilon_1\colon (x,y) \to (-y,x),$$
$$\varepsilon_2\colon (x,y) \to (y,x), \qquad (x,y) \in V \oplus \overline{V}$$

define antilinear maps that give $V \oplus \overline{V}$ both a real and quaternionic structure, relative to either of which $V \oplus \overline{V}$ may be irreducible. However, a self-conjugate complex representation V of a compact group G, irreducible in the complex sense, has either a real or a quaternionic structure, but not both.

A basic fact is that any compact Lie group G possesses a faithful linear complex representation V; this amounts to asserting that G is a Lie subgroup of $U(n)$ for some n sufficiently large. The set of all irreducible representations with increasing highest weights can then be realized as submodules of tensor products of the form

$$\Big(\bigotimes^p V\Big)\otimes\Big(\bigotimes^q \overline{V}\Big),$$

where $\overline{V} \cong V^*$. This is a practical version of the Peter-Weyl theorem, and the reader is invited to get a feeling for the representation theory of a specific group by attempting to decompose tensor products of two or three low-dimensional faithful representations.

Returning to (6.2), more can now be said about the set of weights α that arise in a given irreducible representation V. Certainly, this set is a union of orbits of the Weyl group. If α is a weight of V, and β a root, then $\mathfrak{g}_\beta(V_\alpha)$ is either zero or contained in the weight space $V_{\beta+\alpha}$. There is a partial ordering defined on the set of weights by $\alpha \preceq \beta$ if and only if $\alpha-\beta$ is a sum of positive roots, or zero. One can associate to any irreducible representation of a compact Lie group G its *highest weight* $\gamma \in \overline{\Upsilon}$, which occurs with multiplicity one; each remaining weight has the form $\gamma-\alpha$, where α is a sum of positive roots.

6.1 Theorem [Wey] *The mapping $V \mapsto \gamma$ sets up a bijective correspondence between isomorphism classes of irreducible complex representations of G and dominant weights of G.*

The stabilizer of the highest weight space V_γ, thought of as an element of the complex projective space $\mathbb{C}P(V(\gamma))$, is a parabolic subgroup P of $G_{\mathbb{C}}$, and there is a holomorphic embedding i of $G_{\mathbb{C}}/P$ into the projective space. The pullback $L = i^*\mathcal{O}(1)$ of the hyperplane line bundle on $\mathbb{C}P(V(\gamma))$ may be identifed with the line bundle associated to the representation $\rho\colon P \to \operatorname{Aut} V_\gamma^*$; in compact language,

$$\begin{array}{ccc} L = G\times_\rho V_\gamma^* & \longrightarrow & \mathcal{O}(1) \\ \big\downarrow & & \big\downarrow \\ G/C(\gamma) & \stackrel{i}{\hookrightarrow} & \mathbb{C}P(V(\gamma)) \end{array} \qquad (6.7)$$

The space $H^0(G/C(\gamma),\mathcal{O}(L))$ of holomorphic sections of L is naturally isomorphic to $V(\gamma)^*$; this is the *Borel-Weil theorem* [Se].

Now for some examples. The isotropy representation $[\lambda_0^2]$ of the symmetric space $SU(4)/Sp(2)$ (see (5.13)) defines a homomorphism $\rho\colon Sp(2) \to O(5)$ between Lie groups, that induces an isomorphism $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$ of Lie algebras. Since $Sp(2)$ is a connected group, the image of ρ equals $SO(5)$. Moreover, $[\lambda_0^2]$ is the standard representation of $SO(5)$, and the symmetric Lie algebras

$$\begin{aligned} \mathfrak{su}(4) &= \mathfrak{sp}(2) \oplus [\lambda_0^2], \\ \mathfrak{so}(6) &= \mathfrak{so}(5) \oplus \Lambda^1 \end{aligned} \tag{6.8}$$

coincide. The maximal Lie subgroup $Sp(1) \times Sp(1)$ of $Sp(2)$ fixes a vector in $[\lambda_0^2]$. It follows that $\rho(Sp(1) \times Sp(1)) = SO(4)$, and there are identical symmetric Lie algebras

$$\begin{aligned} \mathfrak{sp}(2) &= \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathbb{R}^4, \\ \mathfrak{so}(5) &= \mathfrak{so}(4) \oplus \Lambda^1. \end{aligned} \tag{6.9}$$

Because the kernel of ρ equals $\{1, -1\}$, the above discussion yields

6.2 Proposition *There are double coverings*

$$\begin{aligned} &Sp(1) \times Sp(1) \cong Spin(4) \xrightarrow{2:1} SO(4) \\ &Sp(2) \cong Spin(5) \xrightarrow{2:1} SO(5) \\ &SU(4) \cong Spin(6) \xrightarrow{2:1} SO(6). \end{aligned}$$

By definition, the group $Spin(n)$ is the simply-connected double covering of $SO(n)$, $n \geqslant 3$. The first isomorphism, combined with the well-known $Spin(3) \cong SU(2) \cong SO(3)$ yields a commutative diagram of Lie group homomorphisms

$$\begin{array}{ccccc} Spin(4) & \xrightarrow{\cong} & SU(2) & \times & SU(2) \\ \downarrow{\scriptstyle 2:1} & & \downarrow{\scriptstyle 2:1} & & \downarrow{\scriptstyle 2:1} \\ SO(4) & \xrightarrow{2:1} & SO(3) & \times & SO(3), \end{array} \tag{6.10}$$

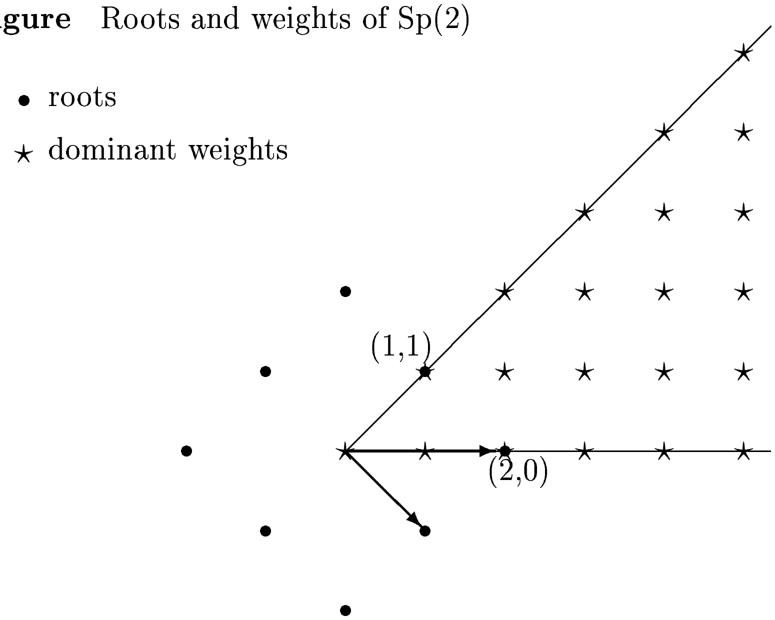
We single out the quaternionic unitary group $Sp(2)$ to give us the simplest and most effective illustration of the theory. In standard coordinates, the roots of $Sp(2)$

can be read off in terms of the isomorphism (5.10) of the complexified Lie algebra with the second symmetric power of the basic $Sp(2)$-module. They are

$$\begin{array}{ll} \boxed{(2,0)} & (1,1) \\ (0,2) & \boxed{(1,\text{-}1)} \\ (0,\text{-}2) & (\text{-}1,1) \\ (\text{-}2,0) & (\text{-}1,\text{-}1), \end{array} \tag{6.11}$$

with a choice of simple roots boxed, the highest $(2,0)$ arising from a simple tensor product $v \otimes v \in \sigma^2$. In this instance, the fundamental Weyl chamber is given by $\Upsilon = \{(a,b) : a > b > 0\}$, the Weyl group acts by permutations and sign changes, and $d = (2,1)$:

6.3 Figure Roots and weights of Sp(2)



The adjoint representation of $Sp(2)$ has non-generic orbit types

$$\frac{Sp(2)}{C(1,0)} = \frac{Sp(2)}{Sp(1) \times U(1)} \cong \frac{SO(5)}{U(2)} \cong \mathbb{C}P(\lambda^1), \tag{6.12}$$

$$\frac{Sp(2)}{C(1,1)} = \frac{Sp(2)}{U(2)} \cong \frac{SO(5)}{SO(3) \times SO(2)} \cong Q^3 \subset \mathbb{C}P(\lambda_0^2),$$

corresponding to a choice of dominant weight in either wall of Υ. The first is biholomorphically equivalent to $\mathbb{C}P^3$, but with a smaller isometry group than usual.

The second is the Hermitian symmetric space parametrizing Lagrangian subspaces of $\mathbb{C}^4$ (endowed with its standard complex symplectic form), and coincides with the quadric $Q^3 \subset \mathbb{C}P^4$.

Tensor products

Given two irreducible representations $V = V(\alpha')$, $W = W(\beta')$, the weights of the tensor product $V \otimes W$ are precisely those of the form $\alpha + \beta$, where α is *any* weight of V, and β *any* weight of W. Each member α of the set A of distinct weights of V will occur with a multiplicity $\mathrm{mult}(\alpha)$. The next result implies that the highest weights of the irreducible summands of $V \otimes W$ can be computed from the list $\alpha + \beta$ of weights formed by adding *all* the weights $\alpha \in A$ of V counted with multiplicity to *only* the highest weight β' of W.

6.4 Proposition [Br] *The multiplicities of the irreducible summands in a tensor product are given by* $V(\alpha') \otimes V(\beta') \cong \bigoplus_{\gamma} n_\gamma V(\gamma)$, *where*

$$n_\gamma = \sum_{\substack{\alpha \in A: \\ \sigma(\alpha + \beta' + d) = \gamma + d}} (-1)^\sigma \mathrm{mult}(\alpha).$$

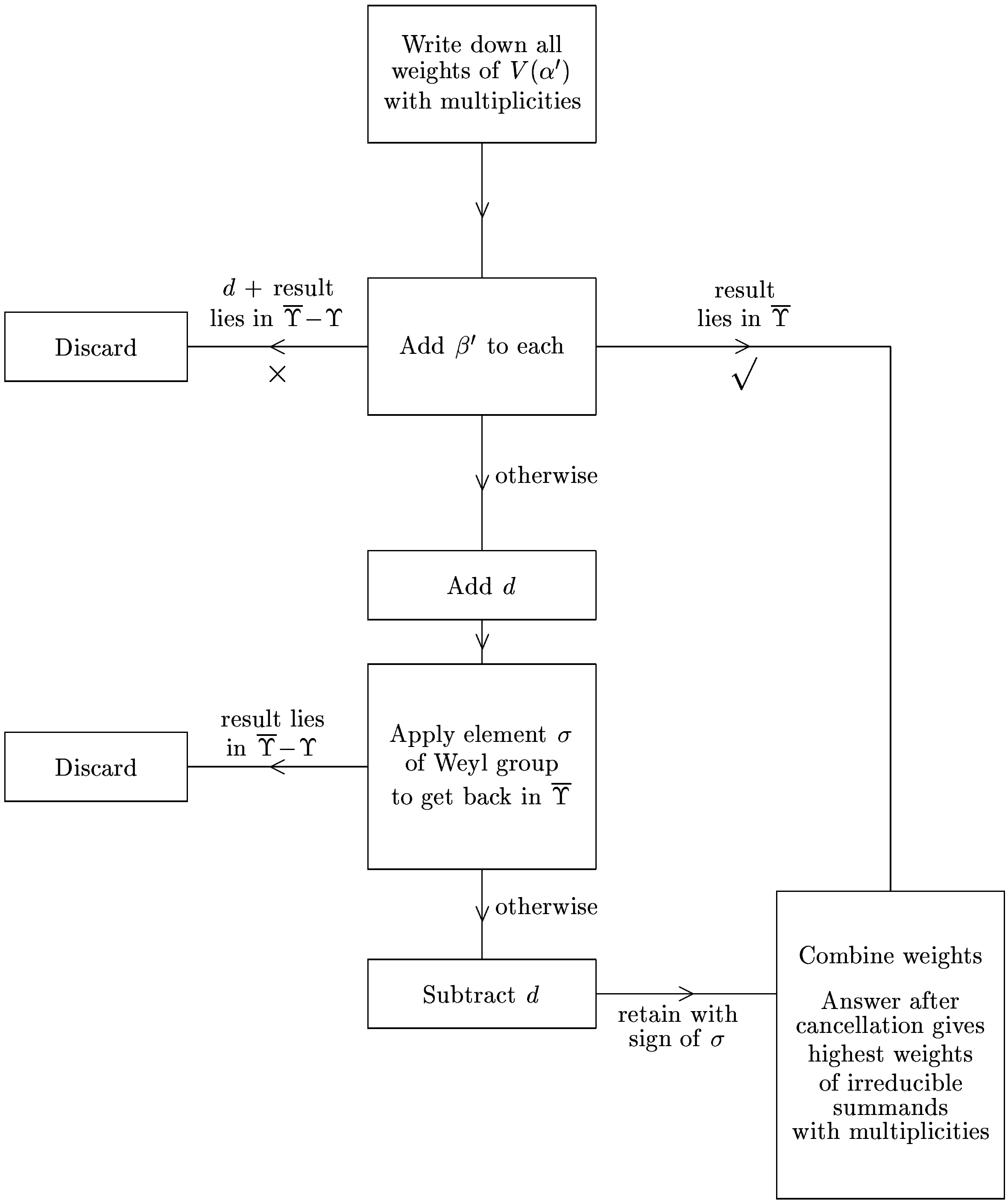
For each summand, σ represents an appropriate element of the Weyl group, as explained below. The result was quoted by Kostant [Ko_2], who finds the multiplicities of all α in a given representation $V(\alpha')$, and there is a related formula for characters is due to Steinberg [St].

The remainder of the chapter will focus on the application of **6.4** to low-dimensional situations. One reason for its power is that the decomposition proceeds in a non-symmetrical fashion; in practice $V(\alpha')$ should be chosen to be the simpler of the two factors. Having said this though, our three examples concern the tensor product of two identical representations, chosen to provide explicit proofs of the decomposition **4.3** of curvature tensors of a Riemannian manifold of dimension $5, 6$ and 7 respectively. The first two examples exploit the identifications **6.2** of $Spin(5)$ and $Spin(6)$, and the resulting "spinorial" descriptions of spaces of tensors, which in a curious way tends to eliminate the amount *skew*-symmetry.

Unfortunately, the group $Spin(n)$ is less useful to us in higher dimensions, not only because of the lack of special isomorphisms, but also because the dimension

($2^{n/2}$ or $2^{(n-1)/2}$) of a faithful representation increases dramatically. In this respect, the case $n = 8$ represents a critical dimension, when tangent vectors and spinors are essentially equivalent (see chapter 12).

6.5 Figure Decomposition of a tensor product in practice



Example 1:
Curvature of a 5-fold or an 8-fold with holonomy $Sp(2)$

The basic representation $\mathbb{C}^4 = \lambda^1$ of $Sp(2)$ determines a *spin representation* of $Spin(5)$, that is, one that does not factor to a representation of $SO(5)$.

6.6 Lemma *There are equivalent definitions*

$$\bigodot^2\mathfrak{sp}(2) \cong [\sigma^4] \oplus \bigodot^2_0[\lambda^2_0] \oplus [\lambda^2_0] \oplus \mathbb{R},$$
$$\bigodot^2\mathfrak{so}(5) \cong W \oplus \Sigma^2_0 \oplus \Lambda^4 \oplus \mathbb{R}.$$

Proof. From (6.11), the highest weight or root of the adjoint representation of $\mathfrak{sp}(2)$ is (2,0). The tensor product of $\mathfrak{sp}(2) = [\sigma^2]$ with itself can now be decomposed by applying **6.5**:

(2,0)	(4,0) ✓			(4,0)
(0,2)	(2,2) ✓			(2,2)
(0,-2)	(2,-2)	(4,-1)	−(4,1)	−(2,0)
(-2,0)	(0,0) ✓			(0,0)
(1,1)	(3,1) ✓			(3,1)
(1,-1)	(3,-1) ×			
(-1,1)	(1,1) ✓			(1,1)
(-1,-1)	(1,-1) ×			
2(0,0)	**2**(2,0) ✓			**2**(2,0).

Using the weights themselves to denote the corresponding $Sp(2)$-modules, we have

$$(2,0)\otimes(2,0) = (4,0) \oplus (2,2) \oplus (0,0) \oplus (3,1) \oplus (1,1) \oplus (2,0), \tag{6.13}$$

and it remains to identify the summands. There is a simpler decomposition

$$(1,1)\otimes(1,1) = (2,2) \oplus (2,0) \oplus (0,0),$$

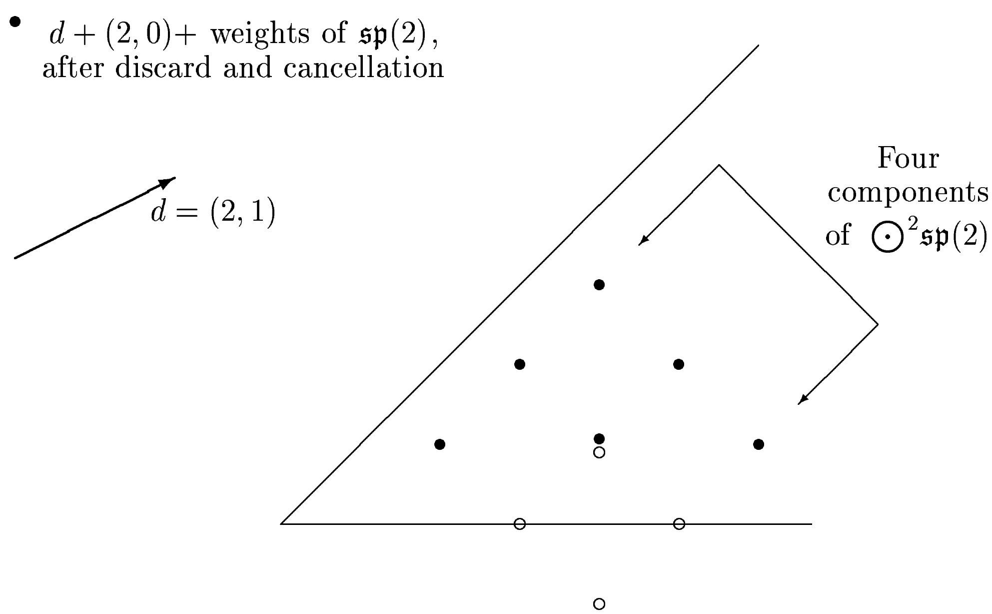
and since $(1,1) = \lambda^2_0$, we must have $(2,2) = \bigodot^2_0(\lambda^2_0)$. The highest weight $(4,0)$ in (6.13) corresponds to the fourth symmetric power $\bigodot^4(1,0) = \sigma^4$, which obviously lies in $\bigodot^2(2,0)$, whereas the weight $(2,0)$ corresponds to the copy of $\mathfrak{sp}(2)$ in $\bigwedge^2(\mathfrak{sp}(2))$ coming from the Lie bracket. A dimension count now confirms that

$$\bigodot^2(2,0) = (4,0) \oplus (2,2) \oplus (1,1) \oplus (0,0),$$

which is equivalent to the first isomorphism in the statement of the lemma. The irreducible components can be reinterpreted in terms of $SO(n)$ to obtain the second isomorphism, which was established more informally in (4.4). □

The submodule $[\sigma^4]$ of $\bigodot^2\mathfrak{sp}(2)$, isomorphic to the space W of Weyl tensors on a 5-manifold, is in fact equal to the space $\mathfrak{R}^{Sp(2)}$ of curvature tensors for the holonomy subgroup $Sp(2)$ of $SO(8)$ on an 8-manifold (see forward to **9.3**).

6.7 Figure Summary of decomposition of $\bigotimes^2\mathfrak{sp}(2)$



Example 2:
Curvature of a 6-fold or an 8-fold with holonomy $SU(4)$

6.8 Lemma *There are equivalent decompositions*

$$\bigodot^2\mathfrak{su}(4) \cong B\oplus[\lambda_0^{2,2}]\oplus[\lambda_0^{1,1}]\oplus\mathbb{R},$$

$$\bigodot^2\mathfrak{so}(6) \cong W\oplus\Sigma_0^2\oplus\Lambda^4\oplus\mathbb{R}.$$

Proof. Consider $U(4)$, whose adjoint representation representation $\mathfrak{u}(4)\otimes_{\mathbb{R}}\mathbb{C}\cong\lambda^{1,1}$ may be understood using (3.5). There are standard coordinates (a,b,c,d) on a maximal torus of $U(4)$, for which the root $(1,0,0,\text{-}1)$ is the highest weight of the irreducible submodule corresponding to $\mathfrak{su}(4)$. The Weyl group consists of the 4! permutations of the coordinates, and a fundamental Weyl chamber is $\Upsilon=\{(a,b,c):a>b>c>d\}$, and $d=(\frac{3}{2},\frac{1}{2},\text{-}\frac{1}{2},\text{-}\frac{3}{2})$. Applying **6.5** gives

(1,0,0,-1)	(2,0,0,-2) ✓			(2,0,0,-2)
(1,0,-1,0)	(2,0,-1,-1) ✓			(2,0,-1,-1)
(1,-1,0,0)	(2,-1,0,-1) ×			
(0,1,0,-1)	(1,1,0,-2) ✓			(1,1,0,-2)
(0,1,-1,0)	(1,1,-1,-1) ✓			(1,1,-1,-1)
(0,0,1,-1)	(1,0,1,-2) ×			
(0,0,-1,1)	(1,0,-1,0) ×			
(0,-1,1,0)	(1,-1,1,-1)	$(\frac{5}{2},-\frac{1}{2},\frac{1}{2},-\frac{5}{2})$	$-(\frac{5}{2},\frac{1}{2},-\frac{1}{2},-\frac{5}{2})$	−(1,0,0,-1)
(0,-1,0,1)	(1,-1,0,0) ×			
(-1,1,0,0)	(0,1,0,-1) ×			
(-1,0,1,0)	(0,0,1,-1) ×			
(-1,0,0,1)	(0,0,0,0) ✓			(0,0,0,0)
3(0,0,0,0)	**3**(1,0,0,-1) ✓			**3**(1,0,0,-1)

Combining the summands, and assessing their dimensions gives

$$\bigodot^2(1,0,0,\text{-}1) \cong (2,0,0,\text{-}2) \oplus (1,1,\text{-}1,\text{-}1) \oplus (1,0,0,\text{-}1) \oplus (0,0,0,0),$$
$$\bigwedge^2(1,0,0,\text{-}1) \cong (2,0,\text{-}1,\text{-}1) \oplus (1,1,0,\text{-}2) \oplus (1,0,0,\text{-}1),$$

the first isomorphism being the one sought. □

In particular, the submodule W of $\mathfrak{R}^{SO(6)}$ containing the space of Weyl tensors on a 6-manifold is isomorphic to the space $\mathfrak{R}^{SU(4)}$ containing the Bochner tensor of a Kähler manifold of real dimension 8 (see **4.7**).

Example 3:
Curvature of a 7-fold or an 8-fold with holonomy $Spin(7)$

In order to split up the tensor product of $\mathfrak{so}(n)$ with itself for $n \geqslant 7$, one is forced, in the absence of any further special isomorphisms, to resort to a direct description in terms of the roots of the orthogonal Lie algebra. The problem with with $\mathfrak{so}(n)$, as opposed to $\mathfrak{su}(m)$ or $\mathfrak{sp}(k)$, is that there are two fundamentally different descriptions of the roots, according to the parity of n. Despite this, the form of the decomposition under discussion does not depend on n. The Lie algebras $\mathfrak{b}_m = \mathfrak{so}(2m+1)$, $\mathfrak{d}_m = \mathfrak{so}(2m)$, and $\mathfrak{u}(m)$ all have rank m, and share a maximal abelian subalgebra; here we tackle the case of $\mathfrak{b}_3$.

6.9 Lemma $\bigodot^2\mathfrak{so}(7) \cong W \oplus \Sigma_0^2 \oplus \Lambda^4 \oplus \mathbb{R}$.

Proof. Given a standard basis $\{e^1, \dots, e^7\}$ of $\Lambda^1 = (\mathbb{R}^7)^*$, the Lie algebra $\mathfrak{so}(7) \cong \Lambda^2$ is generated by the elements $e^i \wedge e^j$, $i < j$, with a maximal abelian subalgebra spanned by $e^1 \wedge e^2$, $e^3 \wedge e^4$, $e^5 \wedge e^6$. The roots can be labelled by the eighteen triples obtained by applying all possible permutations and sign changes to $(1,1,0)$ and $(1,0,0)$, which arise from the weight spaces

$$V_{(1,1,0)} = \operatorname{span}\{(e^1 - ie^2) \wedge (e^3 - ie^4)\}, \quad V_{(1,0,0)} = \operatorname{span}\{(e^1 - ie^2) \wedge e^7\}.$$

The fundamental Weyl chamber is then $\Upsilon = \{(a,b,c) : a > b > c\}$, and half the sum of positive roots is $d = (\frac{5}{2}, \frac{3}{2}, \frac{1}{2})$.

Given the practice built up from the two preceding examples, it is convenient to streamline the notation so as to list only those weights that are not immediately discarded. These form the first column below.

(1,1,0)	(2,2,0) ✓			(2,2,0)
(1,0,1)	(2,1,1) ✓			(2,1,1)
(1,-1,0)	(2,0,0) ✓			(2,0,0)
(1,0,-1)	(2,1,-1)	$(\frac{9}{2},\frac{5}{2},-\frac{1}{2})$	$-(\frac{9}{2},\frac{5}{2},\frac{1}{2})$	$-$(2,1,0)
(1,0,0)	(2,1,0) ✓			(2,1,0)
(0,0,1)	(1,1,1) ✓			(1,1,1)
(-1,-1,0)	(0,0,0) ✓			(0,0,0)
(0,-1,-1)	(1,0,-1)	$(\frac{7}{2},\frac{3}{2},-\frac{1}{2})$	$-(\frac{7}{2},\frac{3}{2},\frac{1}{2})$	$-$(1,0,0)
(-1,1,0)	(0,2,0)	$(\frac{5}{2},\frac{7}{2},\frac{1}{2})$	$-(\frac{7}{2},\frac{5}{2},\frac{1}{2})$	$-$(1,1,0)
(0,-1,0)	(1,0,0) ✓			(1,0,0)
(0,0,-1)	(1,1,-1)	$(\frac{7}{2},\frac{5}{2},-\frac{1}{2})$	$-(\frac{7}{2},\frac{5}{2},\frac{1}{2})$	$-$(1,1,0)
3(0,0,0)	**3**(1,1,0) ✓			**3**(1,1,0)

Combining the summands, and assessing their dimensions gives

$$\bigodot^2(1,1,0) \cong (2,2,0) \oplus (2,0,0) \oplus (1,1,1) \oplus (0,0,0),$$
$$\bigwedge^2(1,1,0) \cong (2,1,1) \oplus (1,1,0),$$

the first isomorphism being the one sought. Notice that $(1,1,1)$ is the highest weight of Λ^3, but that in seven dimensions this is isomorphic to Λ^4. In all three examples, the $\bigwedge^2$ decompositions illustrate the isotropy-irreducible nature of $SO(\frac{1}{2}n(n-1))/SO(n)$ (cf. (5.17)). $\square$

It is important to realize that not only does **6.9** remain valid when $\mathfrak{so}(7)$ is replaced by $\mathfrak{so}(2m+1)$, but that the same method can be applied so as to all the higher-dimensional cases simultaneously. In the above proof, there were 14 weights of $\mathfrak{so}(7)$ which did not get discarded immediately, of which 3 (spanning $\mathfrak{t}$) are zero. For *any* $m \geqslant 4$, $\mathfrak{so}(2m+1)$ has a total of $m(2m+1)$ weights, but only $2m+11$ of them do not get discarded immediately, of which m are zero. All but one of these zero weights is cancelled by the $m{-}2$ weights which feature the adjacent pair $\dots,\text{-}1,1,\dots$ in any position except the second and third, and the single weight $(0,\dots,0,\text{-}1)$. Of the remaining 11 weights, $(0,\dots,0,\text{-}1,\text{-}1)$ gets discarded after applying σ in **6.5**, and 4 others cancel in pairs. This leaves a total of 6 weights, 4 of which represent irreducible summands in the symmetric product $\bigodot^2\mathfrak{so}(2m+1)$.

We leave the reader to complete the proof of **4.3** by working through the calculations when the dimension $n = 2m$ is even. This case is actually easier, since the roots of $\mathfrak{so}(2m)$ comprise just one Weyl orbit, and do not include $(1,0,\dots,0)$ etc. Further computations involving representations of $SO(n)$ occur in chapter 10.

The reason that the three spin groups in **6.4** are subgroups of $SO(8)$, as well as $Spin(8)$ stems from the fact that $Spin(7)$ can itself be regarded as a subgroup of $SO(8)$, by means of a faithful representation on $\mathbb{R}^8$. The inclusions

$$Sp(1)\times Sp(1)\subset Sp(2)\subset SU(4)\subset Spin(7)\subset SO(8),$$

then give rise to a wealth of possible geometrical structures on an 8-dimensional manifold. Because $Spin(7)$ turns out also to be a possible holonomy group, **6.9** will be relevant to a description of its ensuing curvature tensor.

To summarize, the highest weight summand housing Weyl tensors in the space $\mathfrak{R}^{SO(n)}$ coincides with the reduced space $\mathfrak{R}^{Spin(n)}$ of curvature tensors on an irreducible 8-dimensional Riemannian manifold with holonomy group $H = Spin(n) \subset SO(8)$, for $n = 5$, 6 and 7 (see **12.6**). In each case the tensors arising in eight dimensions are also Weyl tensors, with zero Ricci contraction. We leave the reader to contemplate these links between curvature tensors in different dimensions, and to decide whether they are purely formal or indicative of actual constructions of metrics on 8-dimensional manifolds.

7 Four Dimensions

We now begin an investigation into the geometries arising from some of the low-dimensional groups discussed in the preceding chapter. First on the agenda is the special orthogonal group $SO(4)$, which is locally isomorphic to the product of $SO(3)$ with itself, in contrast to the simplicity of $SO(n)$ for $n \geqslant 5$. This leads to a splitting of the bundle $\Lambda^2 T^*M$ of 2-forms into two halves $\Lambda^2_+ M$ and $\Lambda^2_- M$, and there is an analogous decomposition of the Weyl curvature tensor, already encountered in **4.8**. The study of metrics which are self-dual and Einstein, which means that their curvature tensor has a particularly simple form, has important generalizations in higher dimensions.

The ensuing theory of self-duality is well known for its far-reaching consequences in the context of the Yang-Mills equations. From our point of view, its richness will become apparent in the study the 7-dimensional total space of the bundle $\Lambda^2_- M$, which will subsequently provide examples of a metric with holonomy group equal to G_2. For the present though, we shall have more to say about the hypersurface of $\Lambda^2_- M$ consisting of elements of unit norm, in the light of twistor theory developed from the Penrose programme.

Two-forms and almost complex structures

Consider the action of the special linear group $SL(4,\mathbb{R})$ on $(\mathbb{R}^4)^* = \Lambda^1$ preserving the volume form

$$\vartheta = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \in \Lambda^4. \tag{7.1}$$

Setting

$$\sigma \wedge \tau = b(\sigma, \tau)\vartheta, \quad \sigma, \tau \in \Lambda^2$$

defines a non-degenerate bilinear form b on Λ^2, diagonalized by the six basis elements

$$\begin{aligned} &dx^1 \wedge dx^2 \pm dx^3 \wedge dx^4, \\ &dx^1 \wedge dx^3 \pm dx^4 \wedge dx^2, \\ &dx^1 \wedge dx^4 \pm dx^2 \wedge dx^3. \end{aligned} \tag{7.2}$$

In fact, if $SO_0(3,3)$ denotes the connected component of the group of transformations of Λ^2 fixing b (there are three other components), there is a double covering

$$SL(4,\mathbb{R}) \longrightarrow SO_0(3,3).$$

The three elements of (7.2) with a plus sign generate a subspace Λ^2_+ on which b is positive definite. The subspace Λ^2_- generated by the remaining three elements equals the orthogonal complement of Λ^2_+ with respect to b. It follows that the set of *all* 3-dimensional subspaces on Λ^2 on which the restriction of b is positive definite can be identified with the symmetric space

$$\frac{SO_0(3,3)}{SO(3)\times SO(3)},$$

which in view of (6.10) is isomorphic to the space

$$\frac{SL(4,\mathbb{R})}{SO(4)} \cong \frac{GL(4,\mathbb{R})}{\mathbb{R}^+\times SO(4)}$$

of all conformal structures on $\mathbb{R}^4$. In other words, Λ^2_+ specifies, up to a scaling, the metric on Λ^1 for which $\{dx^1, dx^2, dx^3, dx^4\}$ is an orthonormal basis.

Conversely, given the orientation and metric g on Λ^1 (and so on Λ^2), there is an $SO(4)$-decomposition

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-, \tag{7.3}$$

where $\Lambda^2_\pm$ are the eigenspaces of the conformally invariant involution $*$ of Λ^2 for which $b(*\sigma,\tau) = g(\sigma,\tau)$. Two-forms lying in Λ^2_+ and Λ^2_- are often called *self-dual* and *anti-self-dual* respectively. From a representation-theoretic point of view, (7.3) is equivalent to the Lie algebra splitting

$$\mathfrak{so}(4) \cong \mathfrak{so}(3)\oplus\mathfrak{so}(3) = \mathfrak{su}(2)\oplus\mathfrak{su}(2) = \mathfrak{sp}(1)\oplus\mathfrak{sp}(1).$$

This presents one with the choice of Lie groups to use as the basis of a description of self-duality, namely $SO(3)$ or $SU(2) = Sp(1)$.

The decomposition (7.3) is also analogous to the splitting of 2-forms into types which occurs on a complex manifold, and there is an intimate relationship between the two situations that is of utmost importance for geometrical implications of the theory. This is readily appreciated by restricting to the subgroup $U(2)$ of $SO(4)$ preserving a non-degenerate 2-form ω; then

$$\Lambda^2 = [\![\lambda^{2,0}]\!] \oplus [\lambda_0^{1,1}] \oplus \mathbb{R}\omega$$

(cf. (3.5)). Modifying slightly the coordinates in (3.1), we may write

$$\omega = \omega^1 = dx^1 \wedge dx^2 - dx^3 \wedge dx^4, \tag{7.4}$$

and the real 2-dimensional subspace $[\![\lambda^{2,0}]\!]$ is spanned by the real components

$$\omega^2 = dx^1 \wedge dx^3 - dx^4 \wedge dx^2, \quad \omega^3 = dx^1 \wedge dx^4 - dx^2 \wedge dx^3 \tag{7.5}$$

of the complex symplectic form

$$(dx^1 - idx^2) \wedge (dx^3 + idx^4) = \omega^2 + i\omega^3. \tag{7.6}$$

The $SO(4)$ and $U(2)$ decompositions of 2-forms are therefore linked by

$$\Lambda^2_+ \;=\; [\lambda_0^{1,1}], \qquad \Lambda^2_- \;=\; [\![\lambda^{2,0}]\!] \oplus \mathbb{R}\omega; \tag{7.7}$$

our choice of orientation stems from the deliberate assumption that $U(2)$ fixes a non-zero element of Λ^2_-.

The orbit $SO(4)/U(2)$ of ω equals the 2-sphere

$$Z = \{a_1\omega_1 + a_2\omega_2 + a_3\omega_3 : a_1{}^2 + a_2{}^2 + a_3{}^2 = 1\}, \tag{7.8}$$

and in the presence of the metric g, each element of $z \in Z$ may be thought of interchangeably as a 2-form or an almost complex structure in accordance with (3.8). Just as the splitting (7.3) depends only on the *conformal* class of g, the same is true of the set of almost complex structures parametrized by (7.8). As z varies, so does its corresponding type decomposition, and we may rewrite (7.7) as

$$\Lambda^2_+ = \bigcap_{z\in Z} [\lambda^{1,1}], \qquad \Lambda^2_- = \sum_{z\in Z} [\![\lambda^{2,0}]\!]. \tag{7.9}$$

The Lie algebraic structure of $\Lambda^2_- \cong \mathfrak{sp}(1)$ becomes transparent when one considers the almost complex structures I_1, I_2, I_3 associated to the basis elements $\omega^1, \omega^2, \omega^3$ of Λ^2_-; they act on $\mathbb{R}^4 \cong \mathbb{H}$ like imaginary quaternions with relations

$$I_1 I_2 = -I_2 I_1 = I_3. \tag{7.10}$$

Associated bundles

Let M be an oriented Riemannian 4-manifold, with corresponding principal $SO(4)$-bundle P. One of the direct consequences of self-duality is the splitting of the Weyl tensor that featured in the description **4.8** of the space $\Re^{SO(4)}$ of curvature tensors of M. This follows from the isomorphism

$$\bigodot^2(\Lambda^2_+ \oplus \Lambda^2_-) \cong \bigodot^2(\Lambda^2_+) \oplus (\Lambda^2_+ \otimes \Lambda^2_-) \oplus \bigodot^2(\Lambda^2_-).$$

Indeed, comparing the right-hand side with **4.3** gives an isomorphism

$$\Sigma^2_0 \cong \Lambda^2_+ \otimes \Lambda^2_-, \tag{7.11}$$

that emphasizes the conformal invariance of these spaces. Then $W = W_+ \oplus W_-$, where

$$W_\pm \cong \bigodot^2_0(\Lambda^2_\pm), \tag{7.12}$$

is the space of traceless symmetric products of elements of $\Lambda^2_\pm$. We shall now study the mechanics of the resulting components of the Weyl tensor, and how they relate to the associated vector bundle $\Lambda^2_- M = P \times_{SO(4)} \Lambda^2_-$ of anti-self-dual 2-forms, defined by the appropriate homomorphism $SO(4) \to SO(3)$.

Let U be an open set of M admitting a section $s \in \Gamma(U, P)$, or equivalently an oriented orthonormal basis of 1-forms at each point. Using these 1-forms in place of the symbols dx^i above, we obtain an orthonormal basis $\{\omega^1, \omega^2, \omega^3\}$ of sections of $\Lambda^2_- M$. From (7.3), the connection 1-form on P has the form

$$\phi = \phi_+ + \phi_-;$$

here $\phi_\pm$ has values in $\mathfrak{so}(3) \subset \mathrm{End}(\Lambda^2_\pm)$. However, the basis s and our preference for Λ^2_- will be constant throughout the subsequent discussion, so we may safely write

$$s^*\phi_- = \begin{pmatrix} 0 & -\psi^1_2 & \psi^3_1 \\ \psi^1_2 & 0 & -\psi^2_3 \\ -\psi^3_1 & \psi^2_3 & 0 \end{pmatrix}.$$

The 1-forms ψ^i_j determine the induced connection on $\Lambda^2_- M$ by means of the formula

$$\nabla\omega^i = \sum_{j=1}^{3} \psi^i_j \otimes \omega^j, \tag{7.13}$$

and its curvature is constituted from the components

$$\Psi^i_j = d\psi^i_j + \psi^i_k \wedge \psi^k_j, \quad 1 \leqslant i,j \leqslant 3, \tag{7.14}$$

of $s^*\Phi_- = s^*(d\phi + \frac{1}{2}[\phi,\phi])_-$. If we set

$$\begin{aligned}\Psi^2_3 &= \Psi^1_+ + \Psi^1_-,\\ \Psi^3_1 &= \Psi^2_+ + \Psi^2_-,\\ \Psi^1_2 &= \Psi^3_+ + \Psi^3_-,\end{aligned}$$

where $\Psi^i_\pm \in \Gamma(U, \Lambda^2_\pm M)$, the following result is deduced from (7.11),(7.12).

7.1 Proposition *(i) the traceless component of the Ricci tensor of M is represented by*

$$\sum_i \Psi^i_+ \otimes \omega^i \in \Gamma(U, \Lambda^2_+ M \otimes \Lambda^2_- M);$$

(ii) if t is the scalar curvature, then

$$\sum_i (\Psi^i_- - \tfrac{1}{12} t\omega^i) \odot \omega^i \in \Gamma(U, \textstyle\bigodot^2_0 (\Lambda^2_- M))$$

is the component of the Weyl tensor in W_-.

Thus, M is Einstein if and only if the Ψ^i_+ vanish, and a neat consequence of (ii) is that $\sum_i \Psi^i_- \wedge \omega^i = -\frac{1}{2} t\vartheta$. When the expression in (ii) vanishes, the Weyl tensor may be said to be *self-dual*, although in practice one simply says that M itself is self-dual, or alternatively *half conformally flat*. The curvature of $\Lambda^2_- M$ assumes a particularly simple form when M is both self-dual and Einstein, for

$$\Psi^1_2 = \tfrac{1}{12} t\omega^3, \quad \Psi^2_3 = \tfrac{1}{12} t\omega^1, \quad \Psi^3_1 = \tfrac{1}{12} t\omega^2, \tag{7.15}$$

A local section of $\Lambda^2_- M$ has the form $\omega = \sum_i a^i \omega^i$, and the dual basis $\{a^1, a^2, a^3\}$ consists of functions or 0-forms on the total space of the vector bundle $\pi\colon \Lambda^2_- M \to M$. Observe that

$$\nabla\omega = \sum_i (da^i + \sum_j a^j \psi^j_i) \otimes \omega^i$$

equals the pullback by ω of $\sum_i b^i \otimes c^i$, where

$$\begin{aligned} b^i &= da^i + \sum_j a^j \pi^* \psi^j_i,\\ c^i &= \pi^* \omega^i. \end{aligned}$$

It follows that the 1-forms b^1, b^2, b^3 annihilate the distribution of horizontal tangent vectors on the total space of $\Lambda^2_- M$. In this context, "horizontal" at a point $x \in \Lambda^2_- M$ means "tangent to a section ω with $\omega(m) = x$ and $(\nabla\omega)|_m = 0$". Equivalently, the horizontal subspace at x is the image of a horizontal subspace $D_p \subset T_p P$ in the principal $SO(4)$-bundle by the differential of the composition

$$\begin{array}{ccccc} P & \longrightarrow & P \times \Lambda^2_- & \longrightarrow & \Lambda^2_- M \\ p & \mapsto & (p, u) & \mapsto & pu = x. \end{array}$$

7.2 Table Dictionary of invariant forms on $\Lambda^2_- M$

Degree	Word	Local expression	Abbreviation
0	aa	$= \sum a^i a^i$	$= \rho$, (radius)^2
1	ab	$= \sum a^i b^i$	$= \frac{1}{2} d\rho$
2	abb	$= 2(a^1 b^2 b^3 + a^2 b^3 b^1 + a^3 b^1 b^2)$	$= \sigma$
	ac	$= \sum a^i c^i$	$= \tau$, tautological 2-form
3	abc	$= a^1 b^2 c^3 + a^2 b^3 c^1 + a^3 b^1 c^2$ $- a^1 b^3 c^2 - a^2 b^1 c^3 - a^3 b^2 c^1$	$= \alpha$
	bbb	$= 6\, b^1 b^2 b^3$	$= \beta$
	bc	$= \sum b^i c^i$	$= d\tau$
4	acc	$= 0$	
	bbc	$= 2(b^1 b^2 c^3 + b^2 b^3 c^1 + b^3 b^1 c^2)$	$= \gamma$
	cc	$= \sum c^i c^i$	$= -6\vartheta$, cf. (7.1)

The invariant forms listed above are all independent of the choice of basis, and so are globally defined on the total space of $\Lambda^2_- M$. They are generated by "words" consisting of two or three of the letters a, b, c, and that correspond to the basis of quadratic and cubic invariants of $SO(3)$. Two letters together stand formally for the dot product; for example $ac = \sum_{i=1}^3 a^i c^i$ is the tautological 2-form, also denoted by τ, whose value at a point $x \in \Lambda^2_- M$ is the pullback of the 2-form that x defines on M. In computing the result, juxtaposition of forms denotes exterior product; the symbol $\wedge$ will frequently be omitted in the sequel. Three adjacent letters denote the determinant or alternating sum, although if two of these are the letter b, a cyclic sum suffices.

The following identities are valid:

$$\begin{aligned}
&3\sigma d\rho = 2\rho\beta,\\
&\sigma\tau = \rho\gamma + \alpha d\rho,\\
&3\gamma d\rho = 2\beta\tau = 6\sigma d\tau,\\
&\tau\gamma = -2\sigma\vartheta = \alpha d\tau,\\
&\gamma d\tau = -2\beta\vartheta.
\end{aligned} \tag{7.16}$$

It remains to calculate the exterior derivatives of the forms σ, α, β, and γ.

7.3 Proposition *(i) M is self-dual if and only if one of the equivalent equations holds:*

$$d\alpha = \gamma - \tfrac{1}{3}t\rho\vartheta,$$

$$d\gamma = \tfrac{1}{3}t\vartheta d\rho.$$

(ii) M is self-dual and Einstein if and only if one of the equivalent equations holds:

$$d\sigma = \beta + \tfrac{1}{6}t(\rho d\tau - \tfrac{1}{2}\tau d\rho),$$

$$d\beta = \tfrac{1}{4}t d\tau d\rho;$$

Proof. The curvature enters through the first of the formulae:

$$\begin{aligned}
db^i &= \sum_j (b^j\pi^*\psi_i^j + a^j\pi^*\Psi_i^j),\\
dc^i &= \sum_j \psi_j^i c^j.
\end{aligned}$$

Because the results sought do not depend on a choice of basis, computations may be simplified by working over a point $m \in M$ for which $\psi_j^i|_m = 0$. For example, at m, we have $da^i = b^i$, and

$$\begin{aligned}
d\alpha - \gamma &= (a^1c^3 - a^3c^1)db^2 + (a^2c^1 - a^1c^2)db^3 + (a^3c^2 - a^2c^3)db^1\\
&= (a^1)^2\pi^*(\Psi_-^2\omega^2 + \Psi_-^3\omega^3) - a^2a^3\pi^*(\Psi_-^2\omega^3 + \Psi_-^3\omega^2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \text{cyclic permutations.}
\end{aligned}$$

If the left-hand side equals $-\frac{1}{3}t\rho\vartheta$, then setting $a^1=0$ shows that $\Psi^2_-\omega^3=\Psi^3_-\omega^2$ equals zero, and using the other terms eventually gives $\Psi^i_-=\frac{1}{12}t\omega^i$. Conversely, the last equations imply that $d\alpha-\gamma=-\frac{1}{3}t\rho\vartheta$.

The proofs of the remaining equations are similar. □

The twistor space

The hypersurface

$$ZM = \{x\in\Lambda^2_- : \rho(x)=1\}$$

is a bundle over M, each fibre of which is the 2-sphere (7.8) parametrizing almost complex structures on the corresponding tangent space to M. Passing from M to ZM effectively accomplishes a reduction of the structure group $SO(4)$ of M to $U(2)$. The equations **7.3** simplify considerably when all the forms are pulled back to ZM, for this amounts to setting $d\rho=0=\beta$.

In conjunction with standard metrics on the fibre and base, the 2-forms $\pm\sigma$ and τ give rise to two invariant almost complex structures J_1, J_2 on ZM, formed by combining the tautological structure on the horizontal vectors with one of two natural complex structures of the real 2-dimensional fibres. The ambiguity is fixed by selecting an orientation of the fibre. If x is an arbitrary point of ZM, choose the local basis $\{\omega^1,\omega^2,\omega^3\}$ so that x is represented by the 2-form $\omega^1(m)=e^1e^2-e^3e^4$, where $\{e^1,e^2,e^3,e^4\}$ is an oriented orthonormal basis of T^*_mM. In other words, $a^2(x)=0=a^3(x)$, and the space of (1,0)-forms at x is spanned by

$$b^2+ib^3,\quad \pi^*(e^1-ie^2),\quad \pi^*(e^3+ie^4)\qquad \text{for } J_1, \tag{7.17}$$

and

$$b^2-ib^3,\quad \pi^*(e^1-ie^2),\quad \pi^*(e^3+ie^4)\qquad \text{for } J_2. \tag{7.18}$$

7.4 Theorem [AHS] *If M is self-dual, then (ZM,J_1) is a complex manifold.*

Proof. Consider the form

$$b \;=\; -b^1(a^2+ia^1a^3)+b^2(a^1-ia^2a^3)+ib^3(1-(a^3)^2), \tag{7.19}$$

whose value at the point $x \in ZM$ with $a^2 = 0 = a^3$ is the element $b^2 + ib^3$ occurring in (7.17). The fact that $g(b, \sum a^i b^i) = 0$ implies that b is tangent to ZM, and $g(b,b) = 0$ now shows that b has type $(1,0)$ with respect to J_1.

Extend the 1-forms e^i to a basis that is covariant constant at m, so that $de^i|_m = 0$. The value at x of the exterior derivative of b, regarded as a 1-form pulled back to ZM so that $b^1|_x = 0$, is

$$\begin{aligned} db|_x &= db^2 + idb^3 \\ &= \pi^*(\Psi^1_2 + i\Psi^1_3) \\ &= \pi^*(\Psi^3_+ - i\Psi^2_+) - \tfrac{1}{12} it\pi^*(\omega^2 + i\omega^3), \end{aligned} \tag{7.20}$$

with the hypothesis that M is self-dual. The 2-forms $\pi^*\Psi^i_+$ have type (1,1) relative to J_1, and $\pi^*(\omega^2 + i\omega^3)$ has type (2,0) (cf. (7.6),(7.7)). Therefore the exterior derivative of any "vertical" (1,0)-form at x has no (0,2)-component. A very similar argument, based on the forms c^i in place of b^i, and consequently not involving curvature, shows that the exterior derivative of *any* $(1,0)$ form at x has no $(0,2)$ component. Integrability follows from the Nirenberg-Newlander theorem, and the invariant nature of J_1. □

Without any curvature hypotheses, J_1 is characterized by the following property. An almost complex manifold defined on an open set U of M by a section $\omega \in \Gamma(U, ZM)$ is integrable if and only if $\omega(U)$ is a holomorphic submanifold of the almost complex manifold (ZM, J_1). In this context, "holomorphic" simply means that the tangent spaces of $\omega(U)$ are invariant by J_1. Combined with the remarks following (7.8), this shows that the definition of (ZM, J_1) can be made conformally invariant. Thus, M is self-dual if and only if it possesses, in a neighbourhood of each point, an abundant supply of (negatively oriented) complex structures that are compatible with its conformal structure.

When M is self-dual, each fibre $\pi^{-1}(m) \cong S^2$ is a complex projective line with normal bundle $N_m = \mathcal{O}(1) \oplus \mathcal{O}(1)$, invariant by the anti-holomorphic involution or *real structure* ε on ZM induced from multiplication by -1 on $\Lambda^2_- M$. Thus M may be thought of as a special sort of quotient of a complex 3-fold. The complexification of its tangent space $T_m M$ can be identified with the space $H^0(\pi^{-1}(m), \mathcal{O}(N_m))$ of holomorphic sections of the normal bundle, and the conformal structure of M is then recovered from the 1-dimensional kernel of

$$\bigodot^2 H^0(\pi^{-1}(m), \mathcal{O}(N_m)) \longrightarrow H^0(\pi^{-1}(m), \mathcal{O}(\bigodot^2 N_m)).$$

The technique of encoding conformal structure into holomorphic data is due to Penrose, and ZM is often called the *twistor space* of M [PR],[Wel].

The almost complex manifold (ZM, J_2) is *never* integrable, not even if M is flat. In fact, J_2 has altogether more to do with real symplectic geometry, since it is characterized by the fact that $\omega \in \Gamma(U, ZM)$ is a closed form on U if and only if $\omega(U)$ is a holomorphic submanifold of (ZM, J_2). Although such holomorphic *sections* may not exist, this fact is extremely relevant to the study of minimal surfaces in M, or conformal harmonic mappings of Riemann surfaces to M, which lift to holomorphic *curves* in (ZM, J_2) [ES].

7.5 Proposition *If M is self-dual and Einstein with $t \neq 0$, then $\omega = -6t^{-1}\sigma + \tau$ is a symplectic form on ZM which calibrates J_1 for $t > 0$ and J_2 for $t < 0$. In the former case, (ZM, J_1) is therefore a Kähler manifold.*

This result is an immediate consequence of **7.3**. Another significant property that the reader may readily verify from (7.18) is that the "canonical" bundle of $(3,0)$-forms relative to J_2 is trivialized by the invariant form

$$d\tau + i\alpha; \tag{7.21}$$

this defines a reduction to $SU(3)$ and means that $c_1(ZM, J_2) = 0$. Many of the above results are included in a precise study of the almost Hermitian geometry of ZM, carried out by Muškarov [Mu].

Two pairs of simply-connected symmetric spaces satisfy the hypotheses of this proposition, namely the sphere S^4 and the complex projective plane $\mathbb{C}P^2$, and their duals. Vanishing of the curvature components in the spaces W_- and Σ_0^2 follows from the absence in these spaces of invariants relative to the respective holonomy groups $SO(4)$ and $U(2)$. Of course, *both* halves of the Weyl tensor of S^4 vanish, the sphere being the only compact simply-connected conformally flat manifold.

The self-duality of S^4 and $\mathbb{C}P^2$ can also be viewed as a consequence of the fact that they constitute the first two of a series of quaternionic symmetric spaces (forming the third column of **5.7**). From this point of view it is easy to identify their twistor spaces; for example the fibration

$$ZS^4 = \frac{SO(5)}{U(2)} \longrightarrow \frac{SO(5)}{SO(4)} = S^4$$

coincides, thanks to (6.12), to the mapping

$$\mathbb{C}P^3 = \frac{Sp(2)}{Sp(1) \times U(1)} \longrightarrow \frac{Sp(2)}{Sp(1) \times Sp(1)} = \mathbb{H}P^1$$

that sends a complex line through the origin in $\mathbb{C}^4$ to its quaternionic span in $\mathbb{H}^2$.

The projection

$$Z\mathbb{C}P^2 = \frac{SU(3)}{S(U(1)\times U(1)\times U(1))} \longrightarrow \frac{SU(3)}{S(U(2)\times U(1))} = \mathbb{C}P^2 \tag{7.22}$$

exhibits the twistor space of $\mathbb{C}P^2$ as the complex 3-dimensional flag manifold. The isotropy representation splits each tangent space of the flag manifold into the direct sum of three complex lines, or rather three real two-dimensional subspaces. In fact, the choices of orientation on each on these subspaces give $Z\mathbb{C}P^2$ a total of $2^3 = 8$ almost complex structures invariant by $SU(3)$, of which $3! = 6$ are integrable. The latter include J_1, relative to which (7.22) is neither holomorphic nor anti-holomorphic, and the (distinct but equivalent) complex structure arising from the identification of $Z\mathbb{C}P^2$ with the projective holomorphic tangent bundle of $\mathbb{C}P^2$. Generalizations of this simple example lead to the twistor theory of symmetric spaces in terms of generalized flag manifolds $G/C(\gamma)$ [BR].

In connection with $\mathbb{C}P^2$, it is appropriate to make some remarks concerning the curvature of a general Kähler surface N, as pictured in **4.8**. It is customary to arrange that the globally defined Kähler form ω be *positively* oriented, so as to reverse the signs of (7.7). This means that the space Λ^2_- is unaffected by the reduction of structure group $SO(4)$ to $U(2)$, leading to the identification

$$W_- = \bigodot{}^2_0(\Lambda^2_-) = \bigodot{}^2_0[\lambda_0^{1,1}] = B$$

of the negative half **7.1**(ii) of the Weyl tensor with the Bochner tensor of N. On the other hand, the positive half of the Weyl tensor of N is completely determined by the scalar curvature t, and consequently N is *anti*-self-dual if and only if t is everywhere zero, a fact exploited by Derdzinski and others [D],[I$_1$],[Boy$_1$].

More on self-dual Einstein manifolds

We shall develop properties of the twistor space ZM of a 4-dimensional self-dual Einstein manifold M. Let L denote the complex line bundle consisting of $(1,0)$-vectors relative to J_1 that are tangent to the fibres. Unless $a^1=0=a^2$, its dual L^{-1} is spanned by the 1-form (7.19), and the computation (7.20) also shows that when M is Einstein, $\overline{\partial}b$ has type $(2,0)$. This means that L^{-1} is a *holomorphic* subbundle of $\lambda^{1,0}M$, and there is a short exact holomorphic sequence

$$0 \to D \longrightarrow T^{1,0}ZM \stackrel{\theta}{\longrightarrow} L \to 0, \tag{7.23}$$

in which $D = \ker\theta$ is a holomorphic distribution transverse to the fibres.

The homomorphism θ may be thought of as a holomorphic 1-form on ZM with values in the bundle L. If s is a holomorphic section of L over an open set $U \subset ZM$, then $\theta = \alpha\otimes s$, for some 1-form α on U. Although $d\theta$ is not defined on ZM, the expression

$$\theta \wedge d\theta \;=\; (\alpha \wedge d\alpha)\otimes s^2$$

makes invariant sense, as its value does not depend the choice of s used in order to compute the derivative. It follows from (7.20) that $\theta \wedge d\theta$ is nowhere-zero when $t \neq 0$, and defines an isomorphism

$$\kappa \;=\; \lambda^{3,0}ZM \;\cong\; L^{-2}. \tag{7.24}$$

A $(2n+1)$-dimensional manifold admitting a line-bundle valued 1-form θ with $\theta \wedge (d\theta)^n$ nowhere-zero is called a *contact* manifold; here we are working in the holomorphic category.

The holomorphic line bundle L is *real* in the sense that $\varepsilon^* L \cong \overline{L}$. In addition, it has a natural unitary structure which gives rise to a canonical connection characterized by the fact that a section is holomorphic if and only if its covariant derivative is bundle-valued form of type $(1,0)$. The curvature of this connection equals the $(1,1)$-form $\overline{\partial}\partial \log\|s\|^2$ (this standard theory is described in more detail at the beginning of the next chapter). Being a "gauge-invariant" 2-form, it must be built up from those of **7.2**, and its pullback to each fibre $\mathbb{C}P^1$ has to coincide with a standard $(1,1)$-form there. The next result is a consequence of these observations.

7.6 Theorem *The twistor space ZM of a self-dual Einstein 4-manifold with non-zero scalar curvature has a complex contact structure, whose line bundle L has curvature proportional to the real symplectic form ω of* **7.5**.

When the scalar curvature t is positive, L is itself a *positive line bundle*. In this case, the above argument also shows that ZM must be Kähler-Einstein; in any case its Ricci tensor measures the curvature of (7.24).

A contact manifold is in many ways an odd-dimensional version of a symplectic manifold, but a more direct link arises from the fact that the standard symplectic structure on the cotangent bundle induces one on the total space of the fundamental line subbundle (in our case L^{-1}) away from its zero section. We shall discuss this a little more in chapter 9, but in the meantime this remark may throw more light on some of the constructions below.

The space $H^0(ZM, \mathcal{O}(T^{1,0}ZM))$ of holomorphic vector fields can be identified with the Lie algebra of all infinitesimal complex automorphisms of ZM, and contains the subalgebra $\mathfrak{g}_{\mathbb{C}}$ of vector fields X for which $[X,Y]$ is a section of the horizontal distribution D whenever Y is. In other words, $\mathfrak{g}_{\mathbb{C}}$ is the space of infinitesimal automorphisms of the contact structure; in our setting it is the complexification of a real Lie algebra defined by the fixed points of the involution ε.

The non-degeneracy of the restriction of $d\theta$ to the horizontal directions implies that θ induces an isomorphism $\mathfrak{g}_{\mathbb{C}} \stackrel{\cong}{\to} H^0(ZM, \mathcal{O}(L))$. For example if $X \in \mathfrak{g}_{\mathbb{C}}$ and $\theta(X) = 0$, then X is horizontal, and the condition on $[X,Y]$ implies that $X = 0$; surjectivity follows from a similar argument [K$_3$, chapter 1]. These observations lead to a split exact sequence

$$0 \to H^0(ZM, \mathcal{O}(D)) \longrightarrow H^0(ZM, \mathcal{O}(T^{1,0}ZM)) \overset{\longleftarrow}{\longrightarrow} H^0(ZM, \mathcal{O}(L)) \to 0.$$

Because L is a positive line bundle, the Kodaira theorem implies that the sections of some sufficiently high power of L will define a projective embedding of ZM. In fact, the morphism

$$ZM \longrightarrow \mathbb{C}P(\mathfrak{g}_{\mathbb{C}}^*)$$

defined by the linear system $|L|$ is itself an embedding in the following two basic situations:

(i) $M = S^4$, $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{sp}(2,\mathbb{C}) \cong \bigodot^2 \mathbb{C}^4$.

The twistor space $ZS^4 = \mathbb{C}P^3$ can be identified with the set of projective classes $[\alpha]$, where α is a non-zero simple tensor product $v \otimes v$, $v \in \mathbb{C}^4$. Such α are characterized by the condition that $\alpha \otimes \alpha$ belong to the 35-dimensional submodule $W_{\mathbb{C}}^*$ of $\bigodot^2 \mathfrak{g}_{\mathbb{C}}^*$ (see **6.6**).

(ii) $M = \mathbb{C}P^2$, $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{sl}(3,\mathbb{C}) \subset \operatorname{End} \mathbb{C}^3$

The twistor space $Z\mathbb{C}P^2$ (7.22) can be identified with the set of projective classes $[\alpha]$, where α is a non-zero endomorphism with square zero. Such a class $[\alpha]$ is determined by the nested subspaces $(\mathrm{Im}\alpha, \ker\alpha)$, and defines a point of the flag manifold. The projection to $\mathbb{C}P^2$ sends the pair $(\mathrm{Im}\alpha, \ker\alpha)$ to the orthogonal complement of $\mathrm{Im}\alpha$ in $\ker\alpha$. The condition $\alpha^2 = 0$ is equivalent to asserting that $\alpha \otimes \alpha$ belongs to the 27-dimensional submodule $B_{\mathbb{C}}^*$ of $\bigodot^2 \mathfrak{g}_{\mathbb{C}}^*$ (see (4.12)).

Basic topological invariants of a 4-manifold include the Euler characteristic $\chi = 2 - 2b^1 + b^2_+ + b^2_-$ and signature $\tau = b^2_+ - b^2_-$, where $b^2_\pm$ denotes the dimension of the space of harmonic (equivalently, closed) 2-forms which are sections of $\Lambda^2_\pm M$. Hirzebruch's signature theorem asserts 3τ equals the evaluation $\langle p_1, [M]\rangle$ of the first Pontrjagin class.

7.7 Theorem *If M is a compact self-dual Einstein 4-manifold with $t > 0$, then the Lie algebra $\mathfrak{g}$ of real contact automorphisms of ZM can be identified with the space of Killing vector fields on M, and has dimension $10 - 2b_2^+$.*

Proof. Since ZM is Kähler-Einstein, Matsushima's theorem [Mat] allows us to identify $\mathfrak{g}$ with the subalgebra of infinitesimal isometries of ZM preserving the horizontal distribution. Since the latter is orthogonal to the fibres, and $\pi\colon ZM \to M$ is a Riemannian submersion (up to a constant), it follows that any element of $\mathfrak{g}$ induces an infinitesimal isometry of M. Conversely, an infinitesimal isometry of M will lift to an element of $\mathfrak{g}$ in the standard way. This bijective correspondence has been made explicit by Nitta and Takeuchi [NiT].

From Serre duality and the vanishing theorem of Kodaira [H], the sheaf cohomology group $H^i(L^n) = H^i(ZM, \mathcal{O}(L^n))$ vanishes whenever $i \geqslant 1$ and $n \geqslant -1$. The dimension $h^0(L^n)$ of the space of holomorphic sections of L^n therefore equals the index

$$\begin{aligned}\sum_{i=0}^{3}(-1)^i \dim H^i(L^n) &= \Big\langle \mathrm{ch}(L^n)\mathrm{td}(T^{1,0}ZM), [ZM]\Big\rangle \\ &= \Big\langle e^{n\ell}e^{\frac{1}{2}c_1(Z)}\hat{A}(TZM), [ZM]\Big\rangle \\ &= \Big\langle e^{n\ell}e^{\ell}\Big(1 - \tfrac{1}{24}p_1([\![L]\!] \oplus \pi^*TM)\Big), [ZM]\Big\rangle \\ &= \Big\langle e^{(n+1)\ell}\Big(1 - \tfrac{1}{24}\ell^2 - \tfrac{1}{24}\pi^*p_1(TM)\Big), [ZM]\Big\rangle,\end{aligned}$$

computed using the Hirzebruch-Riemann-Roch theorem [H]. We have evaluated the Todd class of the *holomorphic* tangent bundle of ZM in terms of the $\hat{A}$ class of its *real* tangent bundle. The symbol ℓ denotes the first Chern class $c_1(L)$, and $[\![L]\!]$ is the real vector bundle underlying L. Now $\langle \ell\pi^*p_1(TM), [ZM]\rangle = -6\tau$, since $\langle \ell, [\mathbb{C}P^1]\rangle = 2$ and a minus sign is needed to to compensate for an orientation reversal. When $n = 0$, we know that $h^0(L^n) = 1$; substituting this in gives $\langle \ell^3, [ZM]\rangle = 8 - 2\tau$ and

$$h^0(L^n) = \tfrac{1}{3}(n+1)\Big[3 + n(n+2)(4-\tau)\Big], \quad n \geqslant -1; \tag{7.25}$$

in particular $h^0(L) = 10 - 2\tau$.

Because M has positive Ricci tensor, Bochner's vanishing theorem implies that $b_1 = 0$. A similar vanishing theorem involving the positive curvature of $\Lambda^2_- M$ gives $b^2_- = 0$, whence the theorem. The fact that $\chi = 2 + \tau$ may also be deduced from an evaluation of the second Chern class of the complex rank 2 vector bundle D whose underlying real bundle is $\pi^* TM$. □

The formula (7.25) is a result about *Fano 3-folds* in disguise. In modern terminology, a Fano manifold is a compact complex manifold whose anti-canonical bundle κ^{-1} is positive, and its *index* is the least positive integer m for which κ^{-1} has an m^{th}-root [Mur]. A well-known characterization of complex projective space $\mathbb{C}P^3$ by Kobayashi and Ochiai [KO] implies that it is the unique Fano 3-fold of index 4. In the present situation, given a self-dual Einstein manifold M with $t > 0$ for which L has a square root, we are allowed to put $n = -\frac{1}{2}$ in (7.25) to make both sides, and consequently τ, equal to zero. In this case, M admits a 10-dimensional group of isometries, and it follows easily that M is isometric to S^4.

A manifold M of dimension n is called *spin* if there exists a principal $Spin(n)$-bundle $\widetilde{P}$ so that $P = \widetilde{P}/\mathbb{Z}$ is a principal $SO(4)$-subbundle of the frame bundle LM. When $n = 4$, this condition is satisfied if and only if the $SO(3)$-structure of $\Lambda^2_- M$ lifts to $SU(2)$, which is equivalent to L having a square root. Lichnerowicz's theorem [L$_2$], mentioned at the end of chapter 4, then provides a direct proof of the vanishing of the Weyl tensor $Weyl_+$ of a compact self-dual spin 4-manifold with $t > 0$. For in the absence of harmonic spinors, the Atiyah-Singer index theorem [AS] implies the vanishing of

$$\begin{aligned} \hat{A}(M) &= -\tfrac{1}{24}\langle p_1, [M]\rangle \\ &= \tfrac{1}{8\pi^2}\int_M \operatorname{tr}(R \wedge R) \\ &= \tfrac{1}{8\pi^2}\int_M (\|Weyl_+\|^2 - \|Weyl_-\|^2)\vartheta, \end{aligned} \tag{7.26}$$

where tr denotes the Killing form on $\mathfrak{so}(4)$, as $Weyl_-$ is already zero.

The classification of compact self-dual Einstein manifolds with $t > 0$ was first settled by Hitchin [H$_2$], and independently by Friedrich and Kurke [FK], by proving that the linear system $|L|$ always gives rise to a *embedding*

$$F \colon ZM \hookrightarrow \mathbb{C}P(\mathfrak{g}^*_{\mathbb{C}}) = \mathbb{C}P^{10-2\tau}.$$

By (7.25), the pullback $F^*\mathcal{O}(2)$ of the square of the hyperplane line bundle has a $(35–8\tau)$-dimensional space of sections, and it is possible to reject the possible images except for (i),(ii) above. Actually, the inequality $|\tau| < \frac{2}{3}\chi$ for Einstein 4-manifolds with $t \neq 0$, derived from (7.26) and the analogous expression

$$\chi = \frac{1}{8\pi^2}\int_M \left(\|Weyl_+\|^2 + \|Weyl_-\|^2 + \tfrac{1}{24}t^2\right)\vartheta,$$

implies that $0 \leqslant \tau \leqslant 3$, and the cases $\tau = 2, 3$ can be ruled out by considering the action of $\mathfrak{g}$ on the harmonic 2-forms resulting from **7.7** [Bes, 13.30]. When $\tau = 1$, the isotropy subgroup must lie in $U(2)$, and it is easy to conclude $M \cong \mathbb{C}P^2$.

7.8 Corollary *A complete self-dual Einstein manifold with scalar curvature $t > 0$ is isometric to S^4 or $\mathbb{C}P^2$, endowed with standard metrics.*

Note that in the presence of a positive definite Ricci tensor, "complete" implies "compact" (by Myers's theorem [My]). Actually, S^4 and $\mathbb{C}P^2$ are the only compact simply-connected 4-manifolds with $\tau \leqslant 1$ admitting a self-dual conformal structure with $t > 0$ [Po_1]. Other topological consequences and interpretations of self-duality have been given by LeBrun [Le_1].

Singular models for the cases $\tau = 2, 3$ are provided by taking M to be the quotient of S^4 or $\mathbb{C}P^2$ by the $\mathbb{Z}_2$-action induced by the geodesic symmetry about a point. For example, when $\mathfrak{g}_\mathbb{C} \cong \mathfrak{so}(4,\mathbb{C}) \cong \bigwedge^2\mathbb{C}^4$ is the complexification of the isotropy Lie algebra of S^4, a point $[\alpha]$ of $F(ZM)$ is such that $\alpha\otimes\alpha$ belongs to the 19-dimensional submodule formed from the spaces (7.11),(7.12), so that $\alpha\cdot\alpha = 0 = \alpha\wedge\alpha$. Thus ZM lies in the singular intersection

$$\{[z_0, z_1, z_2, z_3, z_4, z_5] \in \mathbb{C}P^5 : z_0^2 + z_1^2 + z_2^2 = 0 = z_3^2 + z_4^2 + z_5^2\}$$

of two quadrics, which can be identified with $\mathbb{C}P^3/\mathbb{Z}_2$ induced from the linear map

$$\begin{array}{ccc} \mathbb{C}^2 \oplus \mathbb{C}^2 & \longrightarrow & \bigodot^2\mathbb{C}^2 \oplus \bigodot^2\mathbb{C}^2 \\ (u, v) & \mapsto & (u\otimes u, v\otimes v). \end{array}$$

By finding a twistor space which is a small resolution of a less singular intersection of two quadrics in $\mathbb{C}P^5$, Poon [Po_1] showed that the connected sum $\mathbb{C}P^2\#\mathbb{C}P^2$ admits a self-dual metric with positive scalar curvature. These results have received significant generalizations in [Po_2],[Fl],[DF].

Finally, we consider the case of a self-dual manifold M with zero Ricci tensor. Such a manifold M is sometimes called *half-flat*, although "three-quarters flat" might be more accurate, since only 5 of the 20 curvature dimensions remain. Because (7.15) is satisfied with $t = 0$, the vector bundle $\Lambda^2_- M$ is M is flat. In the context of Yang-Mills theory, they are "gravitational instantons", since their tangent bundle is truly self-dual in the sense that its curvature 2-forms belong entirely to $\Lambda^2_+ M$. Half-flat, and therefore Einstein, metrics also arise from solutions of the *anti-Dirac* or *twistor equation* on an oriented 4-dimensional spin manifold, a fact discussed in [S$_2$],[L$_3$].

If M is half-flat and simply-connected, there exists a global orthonormal basis $\{\omega^1, \omega^2, \omega^3\}$ of covariant constant sections of $\Lambda^2_- M$. The orthogonality of the basis translates into the quaternionic identities (7.10) for the complex structures I_1, I_2, I_3. Then M is *hyperkähler* in the sense that its metric is simultaneously Kähler for each member of the 2-sphere

$$a_1 I_1 + a_2 I_2 + a_3 I_3, \quad a_1^2 + a_2^2 + a_3^2 = 1, \tag{7.27}$$

of complex structures, that determines a holomorphic projection

$$p: ZM \longrightarrow S^2 \cong \mathbb{C}P^1.$$

As a submanifold of ZM, each fibre $p^{-1}(x)$ represents M endowed with the complex structure in question. For example, removing a real line $\pi^{-1}(\infty) \cong \mathbb{C}P^1$ from the twistor space $\mathbb{C}P^3$ of S^4 gives a projection $\mathbb{C}P^3 - \pi^{-1}(\infty) \to \mathbb{C}P^1$, that identifies the twistor space of $\mathbb{R}^4$ with the total space of the holomorphic rank 2 vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ over $\mathbb{C}P^1$. Removing another line $\pi^{-1}(0)$ gives a projection

$$\mathbb{C}P^3 - (\pi^{-1}(0) \cup \pi^{-1}(\infty)) \longrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1 = Q^2$$

to the quadric that parametrizes all orthogonal almost complex structures on $\mathbb{R}^4$.

Another way to look at the hyperkähler 4-manifold M is to fix the Kähler structure defined by ω^1, and observe that the complex symplectic form $\eta = \omega^2 + i\omega^3$ has type $(2,0)$, and trivializes the canonical bundle $\kappa = \lambda^{2,0} M$. One deduces the equivalence of the following conditions for an oriented simply-connected Riemannian 4-manifold:

(i) half-flat, meaning self-dual and zero Ricci tensor,

(ii) hyperkähler, meaning admitting a quaternionic triple of Kähler structures,

(iii) holonomy group H contained in the group $SU(2) = Sp(1)$, whose Lie algebra is isomorphic to Λ^2_+.

A compact complex 2-dimensional manifold with vanishing first Chern class $c_1(M) = -c_1(\kappa)$, and vanishing first Betti number b_1 is called a *K3 surface*. These arise naturally as smooth members of the anti-canonical linear system of divisors on a Fano 3-fold, discussed above. The simplest example is the quartic $\sum_{\alpha=0}^{3}(z^\alpha)^4 = 0$ in $\mathbb{C}P^3$, and one can check that

$$\eta = \left(\frac{z^0}{z^3}\right)^2 d\left(\frac{z^1}{z^0}\right) \wedge d\left(\frac{z^2}{z^0}\right)$$

extends to nowhere-zero closed $(2,0)$ form on this hypersurface.

In terms of classification theory, K3 surfaces are the minimal models for simply-connected surfaces of zero Kodaira dimension. Any two K3 surfaces are diffeomorphic by Kodaira's classification [Ko], and the set of them is parametrized by an irreducible complex space of 20 dimensions, 19 of which arise from the algebraic category. If a finite group of order n acts freely on a K3 surface M, then the quotient has $24/n = \chi \geqslant 2 + |\tau| = 2 + 16/n$, whence $n = 2$ or 4. In fact,

7.9 Theorem [H_1] *A compact 4-manifold M with restricted holonomy group H^0 equal to $SU(2)$ must be a flat torus, a K3 surface, or a quotient of the latter by $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

The case $\pi_1(M) = \mathbb{Z}_2$ corresponds to an Enriques complex surface. A more concrete example of a K3 surface is Kummer's surface obtained by blowing up the sixteen singular points of T/σ, where $T = \mathbb{C}^2/\Lambda$ is a complex torus and σ denotes the involution $(z_1, z_2) \mapsto (-z_1, -z_2)$ of $\mathbb{C}^2$. More explicitly, a punctured neighbourhood of each singular point resembles $(\mathbb{C}^2 - \{0\})/\sigma$, which is naturally isomorphic to the total space of the holomorphic line bundle $\mathcal{O}(-2)$ minus its zero section, and the point can be replaced by the zero section $\mathbb{C}P^1$.

Unfortunately, finding a closed $(2,0)$-form falls a long way short of finding a metric for which it is covariant constant, and the existence of a hyperkähler metric on any K3 surface is a much deeper result that is a consequence of Yau's proof of the Calabi conjecture (see **8.2**), and the existence of some Kähler metric (see [Bea_2] and references therein). More elementary is the existence of a hyperkähler metric on the non-compact total space of the line bundle $\mathcal{O}(-2) \cong \lambda^{1,0}\mathbb{C}P^1$, discovered by Eguchi and Hanson [EH], that we shall discuss in the next chapter. This led to an intuitive description of Yau's metric as sixteen such metrics glued in to a flat torus [P],[H_4], a notion made rigorous by Topiwala [To] who identified the twistor space of a K3 surface.

8 Special Kähler Manifolds

There are two obvious higher-dimensional generalizations of half-flat metrics on four-manifolds, namely Kähler ones with vanishing Ricci tensor, or alternatively those satisfying the stronger condition of being hyperkähler. These two classes are characterized by a holonomy group that is contained in $SU(m)$ or $Sp(k)$ respectively, and coincide when $2m = 4k = 4$.

One of the problems with Kähler geometry is that there are many ways to choose a Kähler metric on a given complex manifold. Yau's proof of the Calabi conjecture singles out a Kähler metric with a preassigned Ricci tensor, and we shall exploit this theorem to exhibit compact manifolds which must admit Ricci-flat metrics. An essential reference for this is Beauville's paper [Bea_1]. The case of $SU(3)$ holonomy has attracted a great deal of attention from physicists in the context of ten-dimensional superstring theory. However, we begin with Calabi's construction, now understood as fundamental, of analogues of the Eguchi-Hanson metric with holonomy $SU(m+1)$ on the total space of a holomorphic line bundle. Similar techniques will be used later on bundles over 4-dimensional manifolds.

By contrast with the Kähler situation, a hyperkähler metric is more or less specified by its associated triple I_1, I_2, I_3 of complex structures. More will be said about this at the end of the next chapter, whereas in the present one we shall exploit the more accurate characterization of hyperkähler metrics by the corresponding 2-sphere of real symplectic forms.

The canonical bundle

Recall from chapter 4 that the Ricci tensor of a Kähler manifold M is essentially the curvature of its canonical bundle $\kappa = \lambda^{m,0}M$. To be more explicit, the covariant derivative of a local section $s \in \Gamma(U, \kappa)$ has the form

$$\nabla s = \psi \otimes s, \tag{8.1}$$

for some connection 1-form ψ. Because s is an $(m,0)$-form on the complex manifold M, we have $\partial s = 0$ and

$$ds = \overline{\partial} s = \psi \wedge s. \tag{8.2}$$

By definition, s is holomorphic if and only if (8.2) vanishes, which is equivalent to the assertion that ψ is a $(1,0)$-form.

The induced Hermitian structure on κ specifies $\|s\|^2 = \langle s, s\rangle$ by

$$m!\, s \wedge \overline{s} \;=\; i^{m^2}\|s\|^2\omega^m, \tag{8.3}$$

and

$$d\,\log(\|s\|^2) \;=\; \frac{1}{\|s\|^2}d\,\langle s, s\rangle \;=\; \psi + \overline{\psi}.$$

When s is holomorphic, $\psi = \partial\log(\|s\|^2)$, and the curvature of κ is represented by the $(1,1)$-form $d\psi$. However, this will be purely imaginary, and the Ricci form $\mathrm{tr}(R)$ defined by (4.14) equals $id\psi$. If $s = dz^1 \wedge \cdots \wedge dz^m$ in local complex coordinates,

$$\mathrm{tr}(R) \;=\; -i\partial\overline{\partial}\,\log(\|s\|^2) \;=\; i\partial\overline{\partial}\,\log\Big(\det(g_{\alpha\overline{\beta}})\Big). \tag{8.4}$$

The manifold M is *Kähler-Einstein* if its Ricci form (8.4) is a multiple of the Kähler form ω at each point.

8.1 Theorem [Ca$_3$] *If M is Kähler-Einstein with scalar curvature $t \neq 0$, there exists a Kähler metric with zero Ricci tensor on a domain of the total space of κ.*

Proof. It is now convenient to take $s \in \Gamma(U, \kappa)$ to be a local *unitary* section, so that $\|s\| = 1$ and ψ is purely imaginary. If a denotes a complex-valued function on U, then the section as is covariant constant at $m \in M$ if and only if $\nabla(as) = (da + a\psi)\otimes s$ vanishes at x. This allows us to regard the coordinate a as a function on the total space of κ; as such it gives rise to a $(1,0)$ form

$$b \;=\; da + a\,\pi^*\psi$$

there, with notation just like **7.2**. Let $\rho = |a|^2$ denote the radius squared, so that $d\rho = a\overline{b} + \overline{a}b$. By assumption, $id\psi = t\omega$, where ω is some positive constant multiple of the Kähler 2-form of M. Then $db = b \wedge \pi^*\psi - iat\pi^*\omega$, whence

$$\widetilde{\omega} \;=\; u\pi^*\omega - t^{-1}u'ib \wedge \overline{b} \tag{8.5}$$

is a globally-defined *closed* (1,1)-form, for any function $u = u(\rho)$, where $u' = du/d\rho$. We shall assume that u and u' are strictly positive functions, so that $\widetilde{\omega}$ determines a Kähler metric.

The exterior derivative

$$d\tau \;=\; da\wedge\pi^*s + a\pi^*(\psi\wedge s) \;=\; b\wedge\pi^*s \tag{8.6}$$

of the tautological form $\tau = a\pi^*s$ on κ is a *closed* or *holomorphic* form of type $(m+1,0)$, whose norm relative to the above Kähler metric is given by

$$\|d\tau\|^2(u(\rho))^m u'(\rho) \;=\; t(m+1),$$

using (8.3). Now $d\tau$ is covariant constant if and only if $\|d\tau\|$ is constant, which is solved by taking $u = (t\rho+\ell)^{\frac{1}{m+1}}$, where ℓ is a constant. Thus (8.5) and some multiple of (8.6) have determined a metric

$$g \;=\; (t\rho+\ell)^{\frac{1}{m+1}}\pi^*g^M + \tfrac{1}{m+1}(t\rho+\ell)^{-\frac{m}{m+1}}\mathrm{Re}(b\otimes\bar{b}), \tag{8.7}$$

whose holonomy is contained in $SU(m+1)$, provided $t\rho+\ell>0$. $\square$

If $t>0$ and ℓ is chosen positive, the metric g is defined everywhere on the total space of the canonical bundle κ, and will be complete if M is. To check this, it suffices to examine the length of geodesics relative to the radial parameter, and the key point is that $\int_0^\infty (t\rho+\ell)^{-m/2(m+1)}d(\rho^{1/2})$ diverges. The basic example comes from the complex projective space $M=\mathbb{C}P^m$, with its standard Fubini-Study metric; in this case the holonomy of κ equals $SU(m+1)$. The canonical bundle $\lambda^{m,0}\mathbb{C}P^m \cong \mathcal{O}(-m-1)$ is dual to the $(m+1)^{\rm st}$ power of the hyperplane line bundle, and away from the zero section resembles $\mathbb{C}^{m+1}/\mathbb{Z}_{m+1}$. As $\rho\to\infty$, (8.7) is locally asymptotic to the Euclidean metric

$$dr^2 + r^2ds^2, \quad r=\rho^{1/2(m+1)}, \tag{8.8}$$

where ds^2 denotes a standard induced metric on the sphere $S^{2m+1}\subset\mathbb{C}^{m+1}$, and r represents distance from the origin in $\mathbb{C}^{m+1}$.

For $k=1$, we obtain the *Eguchi-Hanson metric* over $T^*\mathbb{C}P^1\cong\mathcal{O}(-2)$ described differently in [EH],[EGH]. Another, at first sight unrelated, example of an anti-self-dual Kähler metric is the one in **3.6** with $m=1$, whose scalar curvature turns out to be zero (cf. **4.8**). This observation of Burns was the starting point of a construction of LeBrun [Le_2] of a family of anti-self-dual Kähler metrics on the total space of $\mathcal{O}(-k)$ over $\mathbb{C}P^1$ for all $k>0$ that includes the two examples cited.

Calabi-Yau metrics

Let M be a fixed complex manifold. The formula (8.4) shows that the Ricci curvature of any Kähler metric is determined by its volume form, and the Ricci forms of two Kähler metrics g, g' differ by $-i\partial\overline{\partial}u$ for some real function u. In fact $1/2\pi$ times the Ricci form of any Kähler metric represents the real Chern class

$$c_1(\kappa) = c_1(\lambda^{m,0}M) = -c_1(M).$$

8.2 Theorem $[\mathrm{Y}_2]$ *Let M be a compact Kähler manifold with Kähler form ω, and suppose that $\frac{1}{2\pi}\alpha$ is a closed $(1,1)$-form representing $c_1(M)$. Then there exists a unique Kähler form ω' cohomologous to ω with Ricci form α.*

The interpretation of the Ricci tensor with volume forms, and the usual $\partial\overline{\partial}$ lemma, allows one to restate the theorem as follows. Given a smooth positive function u on M such that $\int_M u\omega^m = \int_M \omega^m$, there exists a smooth function v such that $\widetilde{\omega} = \omega - i\partial\overline{\partial}v$ is a positive $(1,1)$-form satisfying $\widetilde{\omega}^m = u\omega^m$. The uniqueness of v and the formulation of the problem are due to Calabi $[\mathrm{Ca}_2]$. A related theorem due independently to Aubin and Yau asserts that any compact complex manifold whose first Chern class is negative admits a Kähler-Einstein metric, unique up to homothety [Au]. However, we shall require only the corollary of **8.2** that given M, a necessary and sufficient condition for the existence of a Ricci-flat Kähler metric is the vanishing of $c_1(M)$.

Before discussing examples of Kähler manifolds with $c_1 = 0$, we shall deduce some topological consequences of **8.2**. First consider the 4-dimensional situation, with the Euler characteristic $\chi = 2 - 2b_1 + b_2^+ + b_2^-$ and signature $\tau = b_2^+ - b_2^-$. If M is a K3 surface, $\Lambda_+^2 M$ is actually trivialized by a basis of covariant constant forms, and $\chi = 5 + b_2^-$, $\tau = 3 - b_2^-$. Combined with the signature theorem $p_1 = 3\tau$, and standard formulae $p_1 = c_1^2 - 2c_2 = -2c_2$, and $c_2 = \chi$, we obtain

$$b_2^- = 19, \quad \chi = 24, \quad \tau = -16. \tag{8.9}$$

In passing from the case of $SU(2)$ to $SU(3)$ holonomy, one encounters an important difference, which derives from the fact that a compact manifold with holonomy *equal* to $SU(m)$ has no holomorphic $(p,0)$-forms for $1 \leqslant p \leqslant m-1$. This is a consequence of a lemma of Bochner's [BY, page 142], which states that any holomorphic tensor field on a compact Ricci-flat Kähler manifold is necessarily covariant constant, and so $SU(m)$-invariant. In particular, a manifold M with holonomy group equal to $SU(m)$, for $m \geqslant 3$, has no harmonic forms of type $(2,0)$. It follows that M must be projective, for one can approximate the Kähler form by a positive harmonic $(1,1)$-form representing a rational cohomology class.

The remaining Betti numbers on a compact real 6-dimensional manifold with holonomy group equal to $SU(3)$ are then given by $b_1 = 0$, $b_2 = h_0^{1,1}$ and $b_3 = 2h_0^{2,1}$, where $h_0^{p,q}$ denotes the dimension of the space of primitive harmonic forms of type (p,q) (see **3.1**, **4.11**). Thus the Euler characteristic is

$$\chi = 2(1 + h_0^{1,1} - h_0^{2,1}).$$

Given a holonomy reduction in dimensions divisible by four, other topological formulae conspire to estimate the Betti numbers. This is illustrated by

8.3 Theorem *The Betti numbers of a compact* 8*-manifold with holonomy group equal to* $SU(4)$ *satisfy* $b_3 + b_4^+ \geqslant 50$*, and the Euler characteristic is divisible by* 6*.*

Proof. Following the techniques of [S$_1$, section 7], we begin with the formulae

$$\begin{aligned} \frac{1}{45.2^7}\left(7p_1^2 - 4p_2\right) &= 2,\\ \frac{1}{45}(7p_2 - p_1^2) &= \tau, \end{aligned} \tag{8.10}$$

which are two applications of the Atiyah-Singer index theorem. The first expresses the equality of the $\hat{A}$ or Todd genus with the index of the Dirac or 2-step Dolbeault complex

$$\Gamma(M, \lambda^{0,0}M \oplus \lambda^{0,2}M \oplus \lambda^{0,4}M) \xrightarrow{\overline{\partial} + \overline{\partial}^*} \Gamma(M, \lambda^{0,1}M \oplus \lambda^{0,3}M). \tag{8.11}$$

(the Dirac and Dolbeault complexes are equivalent in the presence of the reduction to the *special* unitary group; see (12.2)). Vanishing theorems can be used to show that the only harmonic forms are sections of the trivial bundles $\lambda^{0,0}M$, $\lambda^{0,4}M$, so the index is indeed 2.

The second formula in (8.10) is the Hirzebruch signature theorem, expressing the equality between the L-genus and the index τ of the extension of (8.11) by tensoring both sides with $\bigoplus_{p=0}^{4} \lambda^{p,0}$. The resulting "twisted Dirac complex" is built around the spaces

$$\begin{aligned} \Lambda^4_+ &\cong [\![\lambda^{4,0}]\!] \oplus [\![\lambda^{2,0}]\!] \oplus [\lambda_0^{2,2}] \oplus \mathbb{R},\\ \Lambda^4_- &\cong [\![\lambda_0^{3,1}]\!] \oplus [\lambda_0^{1,1}], \end{aligned} \tag{8.12}$$

whose decompositions are deduced from **3.1** and a little guesswork. Thus,

$$\begin{aligned} b_2 &= 1 + h_0^{1,1}, \\ b_3 &= 2h_0^{2,1}, \\ b_4^+ &= 3 + h_0^{2,2}, \quad b_4^- = 2h_0^{3,1} + h_0^{1,1}. \end{aligned} \tag{8.13}$$

A third equality

$$4p_2 - p_1^2 = 8\chi$$

follows from the equations $p_1 = -2c_2$, $p_2 = 2c_4 + c_2^2$ and $c_4 = \chi$, and is actually valid for any 8-manifold admitting a (merely topological) reduction of structure group to $Sp(2)Sp(1)$, $Spin(7)$ or $SU(4)$, a consequence of computations involving a maximal abelian subalgebra in $\mathfrak{so}(8)$. Combined with (8.10), we obtain

$$\chi = 3\tau - 96. \tag{8.14}$$

Combining this with (8.13) yields

$$\begin{aligned} h_0^{2,2} + 2h_0^{2,1} &= 47 + 3h_0^{1,1} + 4h_0^{3,1}, \\ h_0^{1,1} - h_0^{2,1} + h_0^{3,1} &= \tfrac{1}{6}\chi - 9, \end{aligned}$$

the last equation could also have been found by tensoring (8.11) by $\lambda^{1,0}M$ and computing an index. □

The most straightforward way of exhibiting Kähler manifolds with $c_1 = 0$ is as complete intersections of hypersurfaces in a complex projective space $\mathbb{C}P^m$. Let $f_1, \ldots f_k$ be homogeneous polynomials of degrees $d_1, \ldots d_k$ in the variables $z^0, \ldots, z^m$, so as to define hypersurfaces $M_1, \ldots, M_k$ in $\mathbb{C}P^m$. Provided $df_1 \wedge \cdots \wedge df_k$ is non-zero at all points in the intersection $M = \bigcap_i M_i$, then the latter is a smooth k-codimensional submanifold M of $\mathbb{C}P^m$, and

$$T\mathbb{C}P^m|_M \cong TM \oplus \mathcal{O}(d_1)|_M \oplus \cdots \oplus \mathcal{O}(d_k)|_M.$$

The total Chern class of M is given by

$$c(TM) = (1+x)^{m+1} \prod_{i=1}^{k} (1 + d_i x)^{-1}, \tag{8.15}$$

and $c_1 = 0$ if and only if $\sum_{i=1}^{k} d_i = m+1$. The Euler characteristic

$$\chi = \langle c_{m-k}(M), [M] \rangle$$

of M equals the coefficient of x^{m-k} in (8.15) times $\prod d_i$. Using the notation $(m|d_1,\ldots,d_k)_\chi$, the first few intersections are

$$(3|4)_{24},$$

$$(4|5)_{-200}, \quad (4|3,2)_{24},$$

$$(5|6)_{2610}, \quad (5|4,2)_{-176}, \quad (5|3,3)_{-144}, \quad (5|2,2,2)_{24},$$

$$(6|7)_{39984}, \quad (6|5,2)_{2190}, \quad (6|4,3)_{1476}, \quad (6|3,2,2)_{-144}.$$

New examples can often be created by quotienting out by a finite group. The simplest one consists of the hypersurface $\sum_{\alpha=0}^{p-1}(z^\alpha)^p = 0$ of $\mathbb{C}P^{p-1}$ with p a prime number; this admits a free action by the group generated by $(z^\alpha) \mapsto (\omega^\alpha z^\alpha)$, where ω is a primitive p^{th} root of unity, and $\pi_1(M) = \mathbb{Z}_5$. However, non-simply-connected examples with holonomy group equal to $SU(m)$ can only arise when m is *odd*. For when m is even, the Todd or arithmetic genus

$$\chi(M, \mathcal{O}) = \sum_{p=0}^{m} (-1)^p h^{p,0}$$

is equal to two; this is also true when M is replaced by any covering (necessarily finite by **10.8**), contradicting the multiplicative behaviour of the genus.

Manifolds with holonomy group equal to $SU(3)$, so-called Calabi-Yau spaces, are much favoured by physicists as an ingredient in the compactification of $E_8 \otimes E_8$ superstring theory; see, for example, [Hu]. A key point is that the $SU(3)$ holonomy reduction is characterized by a parallel spinor, which results from the fact that the restriction of the basic representation of $Spin(6) = SU(4)$ to $SU(3)$ contains an invariant (cf. **12.3**). The search for examples with small Euler characteristic has led to various ingenious modifications of the above methods, for example

(i) replacing $\mathbb{C}P^m$ by a product $\mathbb{C}P^{m_1} \times \cdots \mathbb{C}P^{m_r}$ of projective spaces, corresponding to configurations

$$\left(\begin{array}{c|ccc} m_1 & d_{11} & \cdots & d_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ m_r & d_{r1} & \cdots & d_{rk} \end{array} \right), \qquad m_i + 1 = \sum_{j=1}^{k} dij,$$

which have been studied in [Y_3],[GHu].

(ii) replacing $\mathbb{C}P^m$ by a weighted projective space, defined as the quotient of $\mathbb{C}^{m+1} - \{0\}$ by the $\mathbb{C}^*$ action $(z^0, \ldots, z^m) \mapsto (t^{w_0} z^0, \ldots, t^{w_m} z^m)$. In this case the degrees $d_1, \ldots d_k$ of the intersecting hypersurfaces must satisfy $\sum d_i = \sum w_i$, and the intersection must keep clear of the singularities.

(iii) performing a quotient by a finite group with fixed points, and then resolving the singularities. For example an isolated point associated to a singularity of the form $\mathbb{C}^2/\mathbb{Z}_3$ can be blown up by "gluing " in a copy of the complex symplectic manifold $T^*\mathbb{C}P^2$. Yau's theorem will then guarantee the existence of a metric with $SU(3)$ holonomy on the resulting non-singular space. Examples occur in [CHSW],[SW], and we shall be studying a process of this sort for the holonomy group $Sp(2)$ below.

Hyperkähler manifolds

We have been speaking about reductions to $SU(m)$. If $m = 2k$ is even, then, in view of (5.8), it is natural to seek a further reduction to the quaternionic unitary group $Sp(k)$. Since $Sp(k)$ equals the subgroup of $SO(4k)$ that commutes with right multiplication of the imaginary quaternions i, j on $\mathbb{R}^{4k}$, the holonomy group H of a Riemannian manifold is contained in $Sp(k)$ if and only if there exists a pair I_1, I_2 of anti-commuting almost complex structures that are covariant constant. Both I_1 and I_2 give M the structure of a Kähler manifold, the $Sp(k)$-structure being the intersection of the two corresponding $U(2k)$-structures. If we set $I_3 = I_1 I_2$, then we obtain a 2-sphere of complex structures $\sum_{i=1}^3 a^i I_i$ as in (7.27).

Let $\omega^1, \omega^2, \omega^3$ denote the Kähler 2-forms corresponding to I_1, I_2, I_3; in flat space $\mathbb{R}^{4k}$, they are just the imaginary components

$$\begin{aligned} \omega^1 &= \sum_{\alpha=1}^{k} (dx_1^\alpha \wedge dx_2^\alpha - dx_3^\alpha \wedge dx_4^\alpha) \\ \omega^2 &= \sum_{\alpha=1}^{k} (dx_1^\alpha \wedge dx_3^\alpha - dx_4^\alpha \wedge dx_2^\alpha) \\ \omega^3 &= \sum_{\alpha=1}^{k} (dx_1^\alpha \wedge dx_4^\alpha - dx_2^\alpha \wedge dx_3^\alpha) \end{aligned} \tag{8.16}$$

of $\sum_\alpha \overline{dq}^\alpha \otimes dq^\alpha$, generalizing (7.2). As in the 4-dimensional case (7.6), the form $\eta = \omega^2 + i\omega^3$ has type $(2,0)$ relative to I_1. It is covariant constant, closed and holomorphic, and defines a *complex symplectic structure*. Indeed, η^k is a nowhere-zero section of the canonical bundle κ, and determines the underlying $SU(2k)$-structure that reminds us that M is Ricci-flat.

Conversely, suppose that a Kähler metric is given on a $4k$-dimensional manifold. A reduction to hyperkähler is then accomplished by a covariant constant complex symplectic form $\eta \in \Gamma(M, \lambda^{2,0}M)$ such that $L^*(\eta \wedge \overline{\eta})$ is a non-zero multiple of the fixed Kähler form ω^1, where L^* is the natural contraction $\lambda^{2,2} \to \lambda^{1,1}$. This last condition is automatic if M is irreducible, and in any case will apply on each invariant subspace of $T_m M$.

By appealing to Yau's theorem, a much stronger statement may be given which puts less emphasis on the existing metric, namely a *compact* Kählerian manifold M with a complex symplectic form η admits a hyperkähler structure. For η^k trivializes the canonical bundle, and so there exists a Ricci-flat Kähler metric. With respect to this *new* metric η is covariant constant, because this is true for any holomorphic tensor on a compact Ricci-flat Kähler manifold, by Bochner's lemma, already mentioned above. A recent result of Todorov shows that even the assumption that M be Kählerian be dropped.

The next result, though elementary, is in the same spirit; it assumes the algebraic reduction to $Sp(k)$ has been accomplished pointwise, and gives a powerful criterion for the holonomy reduction.

8.4 Lemma [H_5] *The 2-forms $\omega^1, \omega^2, \omega^3$ arising from an $Sp(k)$-structure are all covariant constant if and only if they are all closed.*

Proof. By assumption, M has a Riemannian metric g and almost complex structures I_i which determine the forms ω^i as in (3.8). The identity $I_1 = I_2 I_3$ translates into $\omega^1 = C(\omega^2 \otimes \omega^3)$, where C is an $O(4k)$-equivariant linear mapping, so

$$\nabla\omega^1 = C(\nabla\omega^2 \otimes \omega^3 + \omega^2 \otimes \nabla\omega^3). \tag{8.17}$$

Let V be the module with highest weight $(0, \ldots, 0, \text{-}1, \text{-}2)$ that features in **3.5**, with respect to the subgroup $U(2k)$ leaving I_1 invariant. Then the right-hand side of (8.17) belongs to the space $[\![V]\!] \otimes [\![\lambda^{2,0}]\!]$ which does not contain $[\![V]\!]$ as a real submodule. Since ω^1 is already closed, it must be covariant constant. □

An alternative condition to impose on an $Sp(k)$-structure is that the almost complex structures I_1, I_2, I_3 all be integrable. In this case the manifold is said to be *hypercomplex*; see **9.12**. An analogue of **3.2** replaces $U(m)$ by $Sp(k)$, and the other three groups by $SO(4k)$, $GL(k, \mathbb{H})$, and $Sp(2k, \mathbb{C})$. Just as a Kähler metric arises from compatible symplectic and complex structures, so a *hyper*kähler metric arises from compatible *complex* symplectic and *hyper*complex structures.

The cotangent bundle T^*M of any complex manifold admits a complex symplectic form η equal to the exterior derivative of the tautological 1-form. If $z^1, \ldots, z^m$ are complex coordinates on M, a point of T^*M (we should really write $\lambda^{1,0}M$ to emphasize the complex structure) has the form $\sum_\alpha w_\alpha dz^\alpha$, and

$$\eta = \omega^2 + i\omega^3 = \sum_\alpha dw_\alpha \wedge \pi^* dz^\alpha. \tag{8.18}$$

The holomorphic cotangent bundle of $\mathbb{C}P^1$ is identical to its canonical line bundle, and the above construction provides a generalization, for $m \geqslant 2$, of this special case distinct from (8.6). The methods of **8.1** may be applied to construct a real non-degenerate closed 2-form ω^1 on the total space of $T^*\mathbb{C}P^m$ which combines with (8.18) in order that

8.5 Proposition [Ca$_3$] *The total space of the cotangent bundle $T^*\mathbb{C}P^k$ of complex projective space has a complete metric with holonomy equal to $Sp(k)$.*

The first example, described next, of a compact manifold with holonomy equal to $Sp(k)$ for $k > 1$ was given by Fujiki [Fu], thereby revealing an error in the paper of Bogomolov [Bo$_2$]. Let K denote any K3 surface, and consider the space $\widetilde{M}$ formed by blowing up the diagonal in $K \times K$, which gets replaced by the projective holomorphic tangent bundle Z of K.

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\ \pi\ } & K \times K \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ M & \longrightarrow & (K \times K)/\sigma. \end{array}$$

The quotient M of $\widetilde{M}$ by the involution σ induced from interchange of the factors is non-singular, and $\widetilde{M}$ is a double cover of M branched over Z.

8.6 Theorem *The compact 8-manifold M admits a metric with holonomy group equal to $Sp(2)$.*

Proof. If η denotes the holomorphic symplectic form on the K3 surface K, the form $\widetilde{\xi} = \pi^*(\eta \otimes 1 + 1 \otimes \eta)$ is invariant by σ and therefore passes to a 2-form ξ on the quotient M. Since $\widetilde{\xi}^2 = \widetilde{\xi} \wedge \widetilde{\xi}$ vanishes only on Z, and only to first order there, ξ^2 must be nowhere zero on M. The existence of a hyperkähler metric on M follows from the fact that M is Kählerian (if K is not algebraic one needs to resort to [V]), together with Yau's theorem as explained above.

The Euler characteristic of M equals

$$\begin{aligned}\chi(M) &= \tfrac{1}{2}\Big(\chi(\widetilde{M}) + \chi(Z)\Big)\\ &= \tfrac{1}{2}\Big((\chi(K))^2 + 2\chi(Z) - \chi(K)\Big)\\ &= \tfrac{1}{2}\Big((24)^2 + 2.48 - 24\Big)\\ &= 324,\end{aligned}$$

which precludes it from being reducible. The fact that its holonomy group equals $Sp(2)$ now follows from the classification **10.7**, or the stronger decomposition theorem **10.8**. □

The techniques of **8.3** combine with decompositions of **9.2** to show the following. Any Riemannian 8-manifold with holonomy group equal to $Sp(2)$ has $\chi = 3\tau - 144$, and there exist non-negative integers p, q such that

$$\begin{aligned}b_1 &= 0,\\ b_2 &= 3 + p,\\ b_3 &= 2q,\\ b^4_+ &= 76 + 7p - 2q, \quad b^4_- = 3p.\end{aligned}$$

In particular $b_3 + b_4^+ \geqslant 76$, and $6|\chi$. In fact, the latter is true for any compact Riemannian 8-manifold whose restricted holonomy group H^0 is contained in $SU(4)$, by the calculations of **8.3**.

The manifold M above is simply-connected, and has $b_3 = 0$, and so $(p,q) = (20,0)$. It may be regarded directly as a resolution of the symmetric product $(K \times K)/\sigma$, and as such is the so-called Douady space or Hilbert scheme $\mathrm{Hilb}^2 K$ [Fo]. Theorem **8.6** then exemplifies a general construction, described by Beauville [Bea_1] and Mukai [Mu_1], in which it is possible to form new hyperkähler manifolds out of "symmetric products" of old ones. If we replace K in **8.7** by a flat torus, M becomes the product of a torus and a Kummer surface, but starting from $K \times K \times K$, one can produce a simply-connected 8-manifold with holonomy $Sp(2)$, and $b_2 = 7$. Other examples and interpretations of this construction occur in [AH].

Symplectic reduction

Let N be a submanifold of an ordinary (that is, real) symplectic manifold M, and let $i^*\omega$ denote the pullback of the symplectic form to N. Using the condition $d(i^*\omega) = i^*(d\omega) = 0$, one can show that the distribution $D \subset TN$ defined by

$$D_n = \{X \in T_nN : X \lrcorner i^*\omega = 0\}$$

is integrable. Working locally if necessary, let us suppose that D is tangent to a foliation with a smooth space of leaves $\widetilde{M}$. The fact that the Lie derivative

$$\mathcal{L}_X\omega = X \lrcorner 3d\omega + d(X \lrcorner \omega) = d(X \lrcorner \omega)$$

vanishes whenever the vector field X belongs to D means that $i^*\omega$ is the pullback of a closed 2-form $\widetilde{\omega}$ on $\widetilde{M}$. It is easy to see that $\widetilde{\omega}$ is non-degenerate, and the quotient $(\widetilde{M}, \widetilde{\omega})$ is called a *symplectic reduction* of (M, ω).

In order to repeat this procedure above under more controlled circumstances, suppose now that G is a group of isometries acting on a Kähler manifold M with tensors g, I, ω as in (3.8), such that $g^*\omega = \omega$ for all $g \in G$. The last condition is automatic when M is compact or irreducible (cf. (5.4)). An element of the Lie algebra $\mathfrak{g}$ of G can be regarded as a Killing vector field X on M for which $0 = \mathcal{L}_X\omega = X \lrcorner \omega$, so locally we can write $X \lrcorner \omega = df_X$. If Y is any tangent vector, $Y(f_X) = \omega(X, Y) = -g(IX, Y)$, whence (i) X is itself tangent to the level sets of f_X, and (ii) IX is normal to the level sets.

In the above situation, a *moment mapping* for the action of G on M is an equivariant mapping $\mu: M \to \mathfrak{g}^*$ satisfying

$$X \lrcorner \omega = \langle d\mu, X \rangle,$$

where the right-hand side represents contraction with X, thought of as an element of $\mathfrak{g}$, leaving an ordinary 1-form on M. Inherent in this definition is the fact that any coadjoint orbit (that is, of G on $\mathfrak{g}^*$) has a natural symplectic structure; the skew pairing between tangent vectors at $\xi \in \mathfrak{g}^*$ determined by $A, B \in \mathfrak{g}$ equals $\langle \xi, [A, B] \rangle$. Using some concepts from the cohomology of Lie algebras, one can establish the existence of moment maps under mild assumptions, for example that $H^1(M, \mathbb{Z})$ be zero and that G be compact or semisimple [MW].

Given μ, consider the G-invariant submanifold $N=\mu^{-1}(0)$, whose tangent space at n is the orthogonal complement of $\{I(X|_n):X\in\mathfrak{g}\}$. If G acts freely on $\mu^{-1}(0)$ with Hausdorff quotient $\widetilde{M}=\mu^{-1}(0)/G$, there is a principal bundle

$$\pi\colon\mu^{-1}(0)\longrightarrow\widetilde{M},$$

and a symplectic form $\widetilde{\omega}$ on $\widetilde{M}$ such that $\pi^*\widetilde{\omega}=i^*\omega$. The tangent space $T_x\widetilde{M}$ can then be identified with the space of vectors orthogonal to the fibre directions at any $n\in\pi^{-1}(x)$, and from the resulting equivariant horizontal distribution, $\widetilde{M}$ acquires a Riemannian metric $\widetilde{g}$ and almost complex structure $\widetilde{I}$, both compatible with $\widetilde{\omega}$. One can verify that $\widetilde{\omega}$ is parallel with respect to $\widetilde{g}$, so that $\widetilde{M}$ is a Kähler manifold.

The simplest example of this construction concerns the action of the circle group $U(1)$ consisting of multiplication by e^{it} on $\mathbb{C}^{m+1}=\mathbb{R}^{2m+2}$, endowed with its standard Kähler form (3.6). It is easy to see that the square $\mu=\sum_{\alpha=1}^{m}|z^\alpha|^2$ of the radius is a moment function, and as $U(1)$ is abelian, we may apply the above construction to the inverse image of a non-zero value in $\mathfrak{g}^*$. The resulting quotient

$$\frac{\mu^{-1}(1)}{U(1)}=\frac{S^{2m+1}}{S^1}\cong\mathbb{C}P^m$$

gives the Fubini-Study metric on complex projective space. The whole procedure is just a real counterpart of the description of $\mathbb{C}P^m$ as the complex quotient parametrizing the generic $\mathbb{C}^*$-orbits of $\mathbb{C}^{m+1}$.

The discovery that symplectic reduction adapts naturally to a hyperkähler context was made by Hitchin, Karlhede, Lindström and Roček [HKLR]. Let G be a group of isometries of a hyperkähler manifold M preserving the holonomy bundle, so that $g^*\omega^i=\omega^i$ for $i=1,2,3$. We shall suppose that there exist moment maps μ^i for each of the three symplectic forms ω^i, or more succinctly, a mapping

$$\mu_q=\mu^1 i+\mu^2 j+\mu^3 k:M\longrightarrow\mathfrak{g}^*\otimes\mathrm{Im}\,\mathbb{H}=\mathfrak{g}^*\otimes\mathfrak{sp}(1).$$

This time the tangent space to μ_q at n is the orthogonal complement of the span of the vectors $I_r(X|_n)$, $r=1,2,3$, $X\in\mathfrak{g}$.

8.7 Theorem [HKLR] *If the quotient $\mu_q^{-1}(0)/G$ is a manifold, then its induced Riemannian metric is hyperkähler.*

Proof. Briefly, this may be accomplished by emphasizing the role of the complex symplectic form $\omega^2+i\omega^2$. The latter has type $(2,0)$ with respect to the complex structure I_1 defining ω_1, which implies that the function

$$\mu_c=\mu^2+i\mu^3\colon M\longrightarrow\mathfrak{g}^*\otimes_{\mathbb{R}}\mathbb{C}$$

is holomorphic. Then $\mu_c^{-1}(0)$ is a Kähler submanifold of M admitting a real moment mapping μ^1, and one can form the quotient

$$\frac{(\mu^1)^{-1}(0)\cap\mu_c^{-1}(0)}{G} = \frac{\mu_q^{-1}(0)}{G}.$$

From above, this will be Kähler, but we can make do with the weaker assertion that it admits a real symplectic form $\widetilde{\omega}^1$ compatible with the induced metric. For, repeating the argument with I_2, I_3 gives two more symplectic forms $\widetilde{\omega}^2, \widetilde{\omega}^3$, and the result follows from **8.4**. $\square$

This time the basic example involves the action of $U(1)$ given by left multiplication by e^{it} on $\mathbb{H}^{k+1}$, whose flat hyperkähler structure is defined by right quaternion multiplication. Here i is some fixed unit quaternion, and the corresponding infinitesimal isometry X satisfies

$$\begin{aligned} X\lrcorner\Big(\sum_{\alpha=0}^{k}\overline{dq}^\alpha\wedge dq^\alpha\Big) &= \sum\Big(\overline{iq^\alpha}dq^\alpha - d\overline{q}^\alpha.iq^\alpha\Big)\\ &= -d\Big(\sum\overline{q}^\alpha iq^\alpha\Big).\end{aligned}$$

The moment mapping is

$$\begin{aligned}\mu_q &= -\sum_{\alpha=0}^{k}\overline{q}^\alpha iq^\alpha\\ &= -\sum(\overline{z}^\alpha-\overline{w}^\alpha j)i(z^\alpha+jw^\alpha)\\ &= i\sum(|w^\alpha|^2-|z^\alpha|^2)-2k\sum z^\alpha w^\alpha,\end{aligned}$$

where $U(1)$ acts on the complex coordinates by $(z_\alpha,w_\alpha)\mapsto(e^{it}z_\alpha,e^{-it}w_\alpha)$.

In general, it is allowable to consider the inverse image by the moment mapping of a point whose components lie in the dual of the centre of $\mathfrak{g}$. Complexifying the above $U(1)$ action yields a complex quotient

$$\frac{\mu_q^{-1}(-i)}{U(1)} \cong \frac{\{(z_\alpha,w_\alpha)\in\mathbb{C}^{k+1}\oplus\mathbb{C}^{k+1}:\sum z_\alpha w_\alpha=0,\ (z_\alpha)\neq 0\}}{\mathbb{C}^*},$$

that can be identified with the holomorphic cotangent bundle $T^*\mathbb{C}P^k$. General principles imply that the induced metric is complete, so in this way a proof of **8.5** is obtained. A modification starting with the action of $U(m)$ on $\mathbb{H}^{p(p+q)}$ produces hyperkähler metrics on the cotangent space of the Grassmannian $Gr_p(\mathbb{C}^{p+q})$ [LR].

Similar constructions produce many examples of previously known hyperkähler metrics, particularly in four dimensions.

The standard complex structure of the line bundle $T^*\mathbb{C}P^1 = \mathcal{O}(-2)$ defines a resolution of the singular space $\mathbb{C}^2/\mathbb{Z}_2$, in which the origin of $\mathbb{C}^2$ has been blown up to the zero section $i_0(\mathbb{C}P^1)$. This resolution fits in with deformations of the singular space to form a $\mathbb{C}P^1$-family of complex structures on $T^*\mathbb{C}P^1$, which determines its hyperkähler structure. More generally, for any finite subgroup Γ of $SU(2)$ the singularity $\mathbb{C}^2/\Gamma$ has a minimal resolution described by Brieskorn, which Kronheimer [Kr_2] exhibits as a hyperkähler quotient of the form

$$M = \frac{\mu_q^{-1}(\xi)}{U}, \quad \xi \in \mathfrak{t}^*\otimes\mathfrak{sp}(1),$$

where U is essentially a product of unitary groups, and the Lie algebra $\mathfrak{t}$ of the centre of U is identified with the maximal abelain subalgebra of one of root systems $\mathfrak{a}_n$, $\mathfrak{d}_n$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ associated to the set of irreducible representations of Γ. Provided the components of ξ in $\mathfrak{h}$ are regular, the metric with $SU(2)$-holonomy on M is complete, and asymptotically locally Euclidean or "ALE" (cf. (8.8)). The singularity itself corresponds to taking $\xi = 0$, and the construction yields the expected number of parameters of such ALE metrics, namely $3\dim\mathfrak{t} - 3$ [GHa],[GP]. The A_n case, which corresponds to $\Gamma = \mathbb{Z}_{n+1}$ cyclic was studied previously by Hitchin [H_2] and gives rise to the so-called multi-Eguchi-Hanson metrics.

The power of **8.7** becomes even more apparent when it is applied in infinite-dimensional situations, in particular to the moment mapping defined by Atiyah and Bott [AB] on the space of connections. Let P is a principal G-bundle over a real $2m$-dimensional symplectic manifold M, where G is a compact semisimple Lie group. By fixing one G-connection, we identify the set of all of them with the space

$$\mathcal{A} = \Gamma(M, \mathrm{Ad}\,P\otimes T^*M)$$

of 1-forms on M with values in the adjoint bundle $\mathrm{Ad}\,P = P \times_G \mathfrak{g}$ (see **1.6**). A symplectic form ω on M furnishes $\mathcal{A}$ with a symplectic form

$$(A, B) = \int_M \mathrm{tr}(A \wedge B) \wedge \omega^{m-1}, \quad A, B \in \mathcal{A},$$

where tr denotes an invariant trace on the Lie algebra $\mathfrak{g}$. The infinite-dimensional gauge group $\mathcal{G} = \mathrm{Aut}\,P$ acts on $\mathcal{A}$ preserving the symplectic structure, and the value $\mu(A)$ of the moment mapping corresponding at $A \in \mathcal{A}$ can be identified with the $2m$-form $F_A \wedge \omega^{m-1}$, where $F_A = dA + \frac{1}{2}[A, A]$ denotes the curvature of the connection A.

Now suppose that M is 4-dimensional, and possessing a triple of symplectic forms $\omega^1, \omega^2, \omega^3 \in \Gamma(M, \Lambda^2_- M)$ that define a hyperkähler metric. The above considerations then lead, at least formally, to a hyperkähler quotient

$$\frac{\mu_q^{-1}(0)}{\mathcal{G}} = \frac{\{A : F_A \in \Gamma(M, \mathrm{Ad}\, P \otimes \Lambda^2_+ M)\}}{\mathcal{G}}$$

consisting of the moduli space of solutions of the Yang-Mills equations with *self-dual curvature*. When M is a higher-dimensional hyperkähler manifold, the equations resulting from the moment mapping are underdetermined, and must be supplemented by extra integrability conditions to obtain Yang-Mills connections [MaS],[Sk].

We describe very briefly some fundamental examples that can be derived from the 4-dimensional picture:

(i) When $G = SU(2)$, there is the "classical" $8k$-dimensional moduli space of self-dual connections on $S^4 = \mathbb{R}^4 \cup \{\infty\}$ with a framing at infinity. Conformal invariance plays a key role in 4-dimensional Yang-Mills theory. The original description by Atiyah, Drinfeld, Hitchin and Manin [ADHM] of this space using quaternionic matrix algebra can be re-interpreted as a *finite*-dimensional hyperkähler quotient construction.

(ii) Other moduli spaces arise from dimensional reduction of the self-dual Yang-Mills equations on $\mathbb{R}^4$; for example, translation-invariance in one direction leads to the Bogomolny equations for monopoles on $\mathbb{R}^3$. Solutions satisfying appropriate boundary conditions constitute a $4k$-dimensional moduli space isomorphic to the space of rational functions $f\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ with $f(\infty) = 0$ [D$_1$]. Corresponding complete hyperkähler metrics have been studied by Atiyah and Hitchin [AH]. The theory of Yang-Mills over a Riemann surface, which results from $\mathbb{R}^2$-invariance, has been developed in [H$_5$].

(iii) Let G be a compact semisimple Lie group, and $\mathfrak{g}$ its Lie algebra. The coadjoint orbits of the complex Lie group $G_{\mathbb{C}}$ on $\mathfrak{g}^*_{\mathbb{C}}$ are naturally complex symplectic, so it is natural to ask whether they admit hyperkähler metrics, as would be the case in a compact setting (see above). Kronheimer [Kr$_3$] has indicated that this is so, and has interpreted special features of the nilpotent orbits, which are in bijective correspondence with conjugacy classes of homomorphisms $\mathfrak{su}(2) \to \mathfrak{g}$. Each orbit is described as a framed moduli space parametrizing certain invariant anti-self-dual connections on $SU(2) \times \mathbb{R}$ extending to S^4.

(iv) On a vector bundle over a projective algebraic surface, Donaldson [D_2] has established a bijective correspondence between stable holomorphic structures and *anti*-self-dual connections. The hyperkähler nature of the relevant moduli spaces also follows from work of Mukai. For example, if K is the K3 surface formed by the intersection of a hyperquadric and a hypercubic in $\mathbb{C}P^4$, then the 8-manifold M of **8.6** is isomorphic to a moduli space of certain stable holomorphic vector bundles on K [Mu_2].

9 Quaternionic Manifolds

The quaternionic unitary group $Sp(k)$ is not a maximal subgroup of $SO(4k)$, since it commutes with the action of the group $Sp(1)$ of unit quaternions. In the present chapter, we shall consider the resulting holonomy group $Sp(k)Sp(1)$, which is only a proper subgroup of $SO(4k)$ when k is greater than one. This determines the class of quaternionic Kähler manifolds, of which hyperkähler manifolds form a subclass. Some constructions of the last two chapters are representative of a more general quaternionic geometry, which in turn provides extra insight into the theory of hyperkähler manifolds.

One reason for giving quaternionic Kähler manifolds serious consideration is Wolf's observation [W_2] that to each compact simple Lie group G, there exists a quaternionic Kähler symmetric space G/H. This theory contrasts favourably with the more sporadic situation of Hermitian symmetric spaces, and the existence of a complex contact manifold fibring over G/H generalizes to the non-symmetric case. The presence of a closed, but highly non-generic, 4-form on a quaternionic Kähler manifold is responsible for both similarities and differences to symplectic geometry.

Quaternionic Kähler manifolds

The group of unit quaternions acting on $\mathbb{H}^k = \mathbb{R}^{4k}$ by right multiplication defines a subgroup $Sp(1)$ of $SO(4k)$. Its centralizer in $SO(4k)$ is precisely the group $Sp(k)$ discussed in chapter 5. The intersection of $Sp(1)$ and $Sp(k)$ consists of plus and minus the identity, and when $k > 1$ the two subgroups generate a proper subgroup

$$Sp(k)Sp(1) \cong Sp(k) \times_{\mathbb{Z}_2} Sp(1) \tag{9.1}$$

of $SO(4k)$, also occasionally denoted $Sp(k) \cdot Sp(1)$.

A *quaternionic Kähler manifold* of dimension $4k \geqslant 8$ is defined to be a Riemannian manifold whose holonomy group is contained in the group $Sp(k)Sp(1)$ [I]. This established terminology can be confusing, because M may not be a Kähler manifold in the ordinary (i.e., complex) sense. In practice, the definition generalizes that of a hyperkähler manifold as follows. Each tangent space still admits a triple of almost complex structures I_1, I_2, I_3 behaving like imaginary quaternions with $I_1 I_2 = -I_2 I_1 = I_3$, but there is no preferred basis, and only the 2-sphere (7.27) makes invariant sense. Consequently all that can be said is that I_1, I_2, I_3 generate a distinguished subbundle of skew-symmetric endomorphisms. The holonomy reduction ensures that this subbundle is preserved by the Levi Civita connection.

It is then possible to make a smooth choice of I_1, I_2, I_3 on a sufficiently small open set, but they cannot be assumed to be *complex* structures. In fact, the existence of two anti-commuting complex structures even locally would lead to a situation much closer to that of a hyperkähler manifold (see **9.12**). However, we *shall* see that the structure of a quaternionic Kähler manifold can always be represented locally by a triple of almost complex structures I_1, I_2, I_3, of which one, I_1 say, is integrable. This fact is by no means obvious from the definitions.

In many ways, the structure group $Sp(k)Sp(1)$ provides an appropriate generalization of four-dimensional Riemannian geometry, which is itself based on the group $SO(4) = Sp(1)Sp(1)$. We shall handle the representations of $Sp(1)$ directly, which amounts to using spinor terminology, rather than passing to the quotient $SO(3)$ as we did chapter 7. For example, the restriction to the subgroup (9.1) of the representation on $\mathbb{R}^{4k}$ defining the tangent or cotangent space has the form

$$\Lambda^1 = [\lambda^1 \otimes \sigma], \tag{9.2}$$

where σ denotes the basic representation of $Sp(1) = SU(2)$ on $\mathbb{C}^2$, and as usual λ^1 denotes the basic representation of $Sp(k)$ on $\mathbb{C}^{2k}$. In this light, the real structure of Λ^1 is inherited from the quaternionic structures of λ^1 and σ.

There is a decomposition of 2-forms

$$\begin{aligned} \mathfrak{so}(4k) \;\cong\; \Lambda^2 \;&\cong\; \textstyle\bigwedge^2[\lambda^1 \otimes \sigma] \\ &\cong\; [\textstyle\bigodot^2 \lambda^1 \otimes \bigwedge^2 \sigma] \oplus [\bigwedge^2 \lambda^1 \otimes \bigodot^2 \sigma] \\ &\cong\; \mathfrak{sp}(k) \oplus [\lambda^2 \otimes \sigma^2] \\ &\cong\; \mathfrak{sp}(k) \oplus \mathfrak{sp}(1) \oplus [\lambda^2_0 \otimes \sigma^2], \end{aligned} \tag{9.3}$$

in which the subspace $\mathfrak{sp}(1)$ is spanned by the basis $\omega^1, \omega^2, \omega^3$ of 2-forms corresponding to I_1, I_2, I_3, expressed in real coordinates in (8.16). In future, $\sigma^r = \bigodot^r \sigma$ will denote the $(r+1)$-dimensional symmetric tensor product. The final summand in (9.3) is the irreducible isotropy representation of the homogeneous space $SO(4k)/Sp(k)Sp(1)$ parametrizing reductions to $Sp(k)Sp(1)$, and which Ziller found missing from Wolf's list [W$_4$],[Z$_2$].

Acting on $\mathbb{R}^{4k}$, the group $Sp(k)Sp(1)$ leaves invariant the 4-form

$$\Omega = \omega^1 \wedge \omega^1 + \omega^2 \wedge \omega^2 + \omega^3 \wedge \omega^3, \tag{9.4}$$

whose top exterior power Ω^k is a constant non-zero multiple of the volume form.

9.1 Lemma *For $k \geqslant 2$, the subgroup of $GL(4k, \mathbb{R})$ preserving Ω is equal to $Sp(k)Sp(1)$.*

Proof. It suffices to show that there is no Lie algebra $\mathfrak{g}$ larger than $\mathfrak{sp}(k) \oplus \mathfrak{sp}(1)$ in $\mathfrak{sl}(4k, \mathbb{R})$ that annihilates Ω. Since $SO(4k)$ does not preserve Ω, the quotient $\mathfrak{g}/(\mathfrak{sp}(k) \oplus \mathfrak{sp}(1))$ must appear as an $Sp(k)Sp(1)$-invariant subspace in

$$\begin{aligned} \frac{\mathfrak{sl}(4k, \mathbb{R})}{\mathfrak{so}(4k)} &\cong \textstyle\bigodot^2_0(\Lambda^1) \\ &\cong (\mathfrak{sp}(k) \otimes \mathfrak{sp}(1)) \oplus [\lambda^2_0], \end{aligned} \tag{9.5}$$

but it is easy to check that neither of the irreducible summands preserve Ω. □

Observe that $Sp(k)Sp(1)$ does lie in the proper subgroup

$$GL(k, \mathbb{H})\mathbb{H}^* \cong GL(k, \mathbb{H}) \times_{\mathbb{Z}_2} Sp(1) \tag{9.6}$$

of $GL(4k, \mathbb{R})$, with Lie algebra

$$\mathfrak{gl}(k, \mathbb{H}) \oplus \mathfrak{sp}(1) \cong \mathfrak{sp}(k) \oplus \mathfrak{sp}(1) \oplus [\lambda^2_0] \oplus \mathbb{R}. \tag{9.7}$$

An immediate consequence of the Clebsch-Gordan formula (12.14) and (9.2) is the existence of $GL(k, \mathbb{H})$-modules V^r_s yielding an analogue

$$\begin{aligned} \Lambda^r &\cong \textstyle\bigwedge^r [\lambda^1 \otimes \sigma] \\ &\cong \bigoplus_{s=0}^{[r/2]} [V^r_s \otimes \sigma^{r-2s}], \quad 0 \leqslant r \leqslant 4k. \end{aligned} \tag{9.8}$$

of the type decomposition (3.3). A complete symmetrization of σ can only be matched by a complete anti-symmetrization of λ^1, so $V^r_0 \cong \lambda^r$. In general, it is not hard to identify V^r_s as an irreducible representation of $GL(k, \mathbb{H})$ defined by an appropriate Young diagram.

The decomposition (9.8), found by Bonan [Bon$_2$], has been refined by Swann [Sw] by expressing each V^i_j as a sum involving the irreducible $Sp(k)$-modules

$$\lambda^r_s = (\underbrace{2, \ldots, 2}_{s}, \underbrace{1, \ldots, 1}_{r-2s}, 0, \ldots, 0), \quad 0 \leqslant r - 2s \leqslant k. \tag{9.9}$$

We abbreviate the real representation $[\lambda^r_s \otimes \sigma^t]$ to to $[\lambda^r_s \sigma^t]$, and declare it to vanish if $r - 2s > k$. The resulting decompositions of exterior forms are listed below for small k.

9.2 Proposition *For $k \geqslant 2$,*

$$\Lambda^1 \cong \underline{[\lambda_0^1\sigma^1]}$$

$$\Lambda^2 \cong \underline{[\lambda_0^2\sigma^2]} \oplus [\sigma^2] \oplus [\lambda_1^2]$$

$$\Lambda^3 \cong [\lambda_0^3\sigma^3] \oplus \underline{[\lambda_0^1\sigma^3]} \oplus [\lambda_1^3\sigma^1] \oplus \underline{\Lambda^1}$$

$$\Lambda^4 \cong [\lambda_0^4\sigma^4] \oplus [\lambda_0^2\sigma^4] \oplus \underline{[\sigma^4]} \oplus [\lambda_1^4\sigma^2] \oplus \underline{[\lambda_1^2\sigma^2]} \oplus \underline{[\lambda_0^2\sigma^2]} \oplus \underline{[\lambda_2^4]} \oplus \underline{[\lambda_0^2]} \oplus \underline{\mathbb{R}}$$

$$\Lambda^5 \cong [\lambda_0^5\sigma^5] \oplus [\lambda_0^3\sigma^5] \oplus [\lambda_0^1\sigma^5] \oplus [\lambda_1^5\sigma^3] \oplus [\lambda_1^3\sigma^3] \oplus [\lambda_2^5\sigma^1] \oplus [\lambda_0^3\sigma^1] \oplus \underline{\Lambda^3};$$

when $k = 2$, only the underlined spaces appear.

The above algebra is transferred to a quaternionic Kähler manifold by supposing now that $\omega^1, \omega^2, \omega^3$ represent the 2-forms corresponding to a local choice of I_1, I_2, I_3. In view of **9.1**, an $Sp(k)Sp(1)$-structure Q on a $4k$-dimensional manifold M is characterized by the existence of a 4-form Ω linearly equivalent at each point to the one in (9.4). Then $\nabla\Omega$ can be identified with the structure function $T_0(p)$ representing the invariant component of the torsion of any connection on the principal bundle Q, and vanishes if and only if M is quaternionic Kähler.

The definition of a quaternionic Kähler manifold is, in some respects, designed to capture properties of the quaternionic projective space $\mathbb{H}P^k$. We choose to define the latter as the quotient of $\mathbb{H}^{k+1}$ by the group $\mathbb{H}^*$ of non-zero quaternions acting by *right* multiplication (cf. (9.6)). A point of $\mathbb{H}P^k$ represents a quaternionic line σ in $\mathbb{H}^{k+1}$. The notation is not accidental, since this line can be identified with the corresponding representation of $Sp(1)$. Indeed, the subgroup of $Sp(k+1)$ stabilizing σ is $Sp(k) \times Sp(1)$, and the orthogonal complement of σ in $\mathbb{H}^{k+1}$ may be identified with the representation λ^1 of $Sp(k)$, so that

$$\mathbb{H}^{k+1} = \sigma \oplus \lambda^1. \tag{9.10}$$

These two summands give rise to vector bundles on $\mathbb{H}P^k$ whose tensor product is isomorphic to the tangent bundle, as in (9.2). On an arbitrary $4k$-dimensional quaternionic Kähler manifold M, the existence of such vector bundles requires the vanishing of a certain mod 2 cohomology class, which Marchiafava and Romani proved is equal to the second Stiefel-Whitney class w_2 when k is odd. By contrast, all quaternionic Kähler manifolds whose dimension is a multiple of 8 are spin manifolds [MR],[S_1].

The above description gives $\mathbb{H}P^k$ the structure of a symmetric space whose holonomy group equals $Sp(k)Sp(1)$, and its curvature is an invariant constituent of the curvature tensor of any quaternionic Kähler manifold.

9.3 Proposition *The spaces of curvature tensors of quaternionic Kähler and hyperkähler manifolds are given respectively by*

$$\mathfrak{R}^{Sp(k)Sp(1)} \cong \bigodot^4[\lambda^1] \oplus \mathbb{R},$$
$$\mathfrak{R}^{Sp(k)} \cong \bigodot^4[\lambda^1].$$

Proof. In arbitrary dimensions, the decomposition **6.6** is still valid, provided $\bigodot^2_0[\lambda^2_0]$ is replaced by the module $[\lambda^4_2]$ of (9.9), and $[\sigma^4]$ is renamed. Hence

$$\begin{aligned}\bigodot^2\Big(\mathfrak{sp}(k) \oplus \mathfrak{sp}(1)\Big) &\cong \bigodot^2\mathfrak{sp}(k) \oplus (\mathfrak{sp}(k)\otimes\mathfrak{sp}(1)) \oplus \bigodot^2\mathfrak{sp}(1)\\ &\cong \bigodot^4[\lambda^1] \oplus [\lambda^4_2] \oplus [\lambda^2_0] \oplus \mathbb{R} \oplus [\lambda^2_1\sigma^2] \oplus [\sigma^4] \oplus \mathbb{R}.\end{aligned}$$

The space $\mathfrak{R}^{Sp(k)Sp(1)}$ of curvature tensors is determined by picking out the summands that are annihilated by the skewing map a of **4.2**; an inspection of **9.2** shows that $\ker a$ must include the highest weight summand $\bigodot^4[\lambda^1]$, not to mention a 1-dimensional subspace that represents the curvature of $\mathbb{H}P^k$. A straightforward, though somewhat laborious, verification with Schur's lemma shows that the remaining spaces inject into Λ^4. For example, in terms of the identification (9.2), the 2-form $\zeta^i = (x^iy^1) \wedge (x^iy^2)$ belongs to $\mathfrak{sp}(k)$, for any $x^i \in \lambda^1$ and suitable $y^1, y^2 \in \sigma$. One can then arrange for

$$a(\zeta^1 \odot \zeta^2) = (x^1y^1) \wedge (x^1y^2) \wedge (x^2y^1) \wedge (x^2y^2)$$

to have non-zero projections to both the subspaces λ^4_2, λ^2_0 of $\Lambda^4\otimes_{\mathbb{R}}\mathbb{C}$.

The formula for $\mathfrak{R}^{Sp(k)}$ now follows from the fact that the curvature of $\mathbb{H}P^k$ cannot lie entirely in $\bigodot^2\mathfrak{sp}(k)$ (see **5.2**). □

9.4 Corollary *Any quaternionic Kähler manifold is Einstein, and its Ricci tensor vanishes if and only if it is locally hyperkähler, i.e. its restricted holonomy group H^0 is a subgroup of $Sp(k)$.*

Of course, the above statements fail when $k = 1$, because Λ^4 is "too small", but in general $\bigodot^4[\lambda^1]$ is the appropriate generalization of the space W_+ housing half the Weyl tensor. In four dimensions it therefore makes sense to *define* quaternionic Kähler to mean self-dual and Einstein.

Twistor spaces and quotients

Complex projective space is the total space of the Hopf fibration

$$\mathbb{C}P^{2k+1} \cong \frac{Sp(k+1)}{Sp(k)\times U(1)} \stackrel{\pi}{\longrightarrow} \frac{Sp(k+1)}{Sp(k)\times Sp(1)} \cong \mathbb{H}P^k, \tag{9.11}$$

which assigns to a complex line in $\mathbb{H}^{k+1}$ its quaternionic span. Each fibre

$$\mathbb{C}P^1 \cong \frac{Sp(1)}{U(1)} \cong \frac{Sp(k)Sp(1)}{Sp(k)U(1)},$$

parametrizes an appropriate reduction of structure, and is a complex submanifold of the total space. This is true, even though π is obviously not holomorphic (in fact $\mathbb{H}P^k$ does not even admit a global almost complex structure).

Each point $z \in \pi^{-1}(x)$ determines an almost complex structure I_z on the real tangent space $T_x\mathbb{H}P^k$ below z. To apply I_z to a vector X, apply the complex structure in $T_z\mathbb{C}P^{2k+1}$ to any lift $\widetilde{X}$, and then project back to $\mathbb{H}P^k$. The family of almost complex structures determined in this manner may be identified with the 2-sphere of unit imaginary quaternions. Indeed, if we regard z as a complex 1-dimensional subspace of the quaternionic line σ, it determines a maximal isotropic subspace $\lambda^1\otimes z$ of the complexified cotangent space $\Lambda^1\otimes_{\mathbb{R}}\mathbb{C}$ to $\mathbb{H}P^k$ at z. This subspace is simply the space of forms of type $(1,0)$ relative to the orthogonal almost complex structure I_z, and defines a corresponding 2-form ω_z, in accordance with the isomorphisms

$$\mathbb{C}P(\sigma) \cong S^2 \subset [\sigma^2] = \mathfrak{sp}(1). \tag{9.12}$$

On any manifold M with $Sp(k)Sp(1)$-structure Q, it is possible consider the associated bundle

$$ZM = Q\times_{Sp(k)Sp(1)}\mathbb{C}P(\sigma)$$

or *twistor space* with fibre the projective line in (9.12). It may be regarded as the sphere bundle sitting inside the distinguished rank 3 subbundle VM of Λ^2T^*M with fibre $\mathfrak{sp}(1)$, just as in the 4-dimensional situation. Remarks above suggest that the special cases **7.6** and (9.11) of the fibration $\pi\colon ZM\to M$ are likely to encapsulate properties of the general case. Indeed,

9.5 Theorem [S$_1$] *If M is a quaternionic Kähler manifold, the associated manifold ZM has a natural complex structure such that*
(i) each fibre is a complex projective line with normal bundle $\mathbb{C}^k \otimes \mathcal{O}(1)$, and the antipodal map on each fibre defines an anti-holomorphic involution of ZM;
(ii) there is a holomorphic distribution D transverse to the fibres defining a 1-form θ with values in the quotient line bundle $L = T^{1,0}ZM/D$;
(iii) if the scalar curvature t is non-zero, $\theta \wedge (d\theta)^k \neq 0$, and $H^0(ZM, \mathcal{O}(L))$ is isomorphic to the complexification of the Lie algebra of Killing vector fields of M.

A local section $s \in \Gamma(U, ZM)$ can itself be regarded as an almost complex structure I on U, which is integrable if and only if s is *holomorphic*, as a submanifold of ZM. This explains an earlier assertion that one of the local almost complex structures I_1, I_2, I_3 defining the quaternionic structure can be chosen to be a complex structure.

There is a direct link between contact and symplectic structures, which is based on existence of a standard symplectic structure on the cotangent bundle. The pull-back $\pi^*\theta$ can be interpreted as a genuine 1-form on the *total space* of the dual line bundle L^{-1} by means of the pairing between L and L^{-1}. Locally if $\theta = \alpha \otimes s$, then we let $\pi^*\theta$ denote $f\pi^*\alpha$, where f is the fibre coordinate of L^{-1} dual to the section s of L. Then

$$\omega = d(\pi^*\theta) = df \wedge \pi^*\alpha + f\pi^* d\alpha$$

is a symplectic form on the principal $\mathbb{C}^*$-bundle consisting of L^{-1} minus its zero section $i_0(ZM)$. In fact ω equals the pullback of the standard symplectic form on the cotangent bundle $\lambda^{1,0}ZM$, and $L^{-1} - i_0(ZM)$ is the *symplectification* of ZM, in the sense of Arnold [Ar].

A function s on a symplectic manifold gives rise to the *Hamiltonian vector field* H_s for which $ds = H_s \lrcorner\, \omega$. The Poisson bracket

$$\{s, t\} = \omega(H_s, H_t) \tag{9.13}$$

of two functions s, t on the total space of L^{-1} then equips the space $H^0(ZM, \mathcal{O}(L))$ of holomorphic sections of L with the structure of a Lie algebra. For the functions s, t are homogeneous of degree one if and only if they correspond to sections s, t of L over M. In this case, H_s and H_t are invariant by the $\mathbb{C}^*$ action on L^{-1}, and $\{s, t\}$ is also homogeneous of degree one and can itself be interpreted as a section of L. This Lie algebra structure identifies the space $\mathfrak{g}_{\mathbb{C}}$ of sections of L with the space of infinitesimal automorphisms of the contact structure, that is those preserving the distribution D.

Property (iii) can now be exemplified with (9.11). From **9.5**(ii), the restriction of L to each fibre $\mathbb{C}P^1$ is isomorphic to its tangent bundle, and it follows that L is the holomorphic line bundle $\mathcal{O}(2)$ over $\mathbb{C}P^{2k+1}$. The space $H^0(\mathbb{C}P^{2k+1}, \mathcal{O}(2)) \cong \bigodot^2(\mathbb{C}^{2k+2})$ of sections is naturally isomorphic to $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}(k+1, \mathbb{C})$, and there is an $Sp(k+1, \mathbb{C})$-equivariant embedding

$$F\colon \mathbb{C}P^{2k+1} \hookrightarrow \mathbb{C}P(\mathfrak{g}_{\mathbb{C}}^*) \cong \mathbb{C}P^{(k+1)(2k+3)}.$$

Just as in case (i) following **7.6**, the image of F consists of the projective orbit of a simple tensor product $v \otimes v$. The latter generates, in suitable coordinates, the space $\mathfrak{g}_\gamma$ corresponding to a highest root γ. As explained in (6.7), the projective orbit can equally be identified with the orbit $Sp(k)/C(\gamma)$ of $\gamma \in \mathfrak{t}$ in $\mathfrak{sp}(k)$. More generally,

9.6 Theorem [W$_2$] *If γ is a highest root in the Lie algebra $\mathfrak{g}$ of a compact simple Lie group G, then the orbit $G/C(\gamma)$ is the twistor space of a quaternionic Kähler symmetric space.*

Proof. Briefly, the details are as follows. In terms of the expression (6.5), define

$$\begin{aligned} \mathfrak{h} &= \mathfrak{t} \oplus [\![\mathfrak{g}_\gamma]\!] \oplus \bigoplus_{\langle\alpha,\gamma\rangle=0} [\![\mathfrak{g}_\alpha]\!], \\ \mathfrak{m} &= \bigoplus_{\langle\alpha,\gamma\rangle=1} [\![\mathfrak{g}_\alpha]\!]. \end{aligned}$$

Because γ is a highest root, the integer $\langle\alpha, \gamma\rangle$ is no greater than 1 unless $\alpha = \gamma$. As a consequence, $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$, and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is an irreducible symmetric Lie algebra. By construction, $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{sp}(1)$, where $\mathfrak{sp}(1)$ denotes the ideal of $\mathfrak{h}$ generated by the root space $\mathfrak{g}_\gamma$, and induces a quaternionic structure on $\mathfrak{m}$ in much the same way as the centre of the isotropy Lie algebra does for a Hermitian symmetric space.

The twistor space $Z(G/H)$ of the corresponding simply-connected symmetric space is the homogeneous space corresponding to the Lie algebra

$$\mathfrak{g} = \Big(\mathfrak{k} \oplus \mathfrak{u}(1)\Big) \oplus \Big([\![\mathfrak{g}_\gamma]\!] \oplus \mathfrak{m}\Big),$$

which is identical to (6.6). □

The classical algebras $\mathfrak{so}(k+4)$, $\mathfrak{su}(k+2)$, $\mathfrak{sp}(k+1)$ give rise to the respective Grassmannians $\widetilde{Gr}_4(\mathbb{R}^{k+4})$, $Gr_2(\mathbb{C}^{k+2})$, $\mathbb{H}P^k$, and the first few "Wolf spaces" of compact type are

$$\widetilde{Gr}_2(\mathbb{C}^3)=\mathbb{C}P^2, \quad \widetilde{Gr}_4(\mathbb{R}^5)=S^4=\mathbb{H}P^1, \quad \frac{G_2}{SO(4)}, \quad \widetilde{Gr}_4(\mathbb{R}^6)=Gr_2(\mathbb{C}^4), \quad \mathbb{H}P^2.$$

The complete list occurs in **5.7**, in which it evident that all the Wolf spaces of compact type apart from $\mathbb{H}P^k$ have $H^2(M,\mathbb{Z}_2)=\mathbb{Z}_2$; this is related to the remarks after (9.10). In fact, $\mathbb{H}P^k$ is characterized by the condition $H^2(M,\mathbb{Z}_2)=0$ amongst all possible complete quaternionic Kähler manifolds with scalar curvature $t>0$. Moreover, any quaternionic Kähler manifold M with $t>0$ has zero odd Betti numbers and positive signature [S_1],[NT]. No complete non-symmetric examples with $t>0$ are known, and in any case must have dimension larger than 8, since an 8-dimensional analogue of **7.8** can be proved by treating the twistor space ZM as a Fano 5-fold of index 3 [PS]. Complete homogeneous examples with $t<0$ were discovered and classified by Alekseevskii [Al_3].

The equivariant embedding F of the twistor space of a Wolf space in $\mathbb{C}P(\mathfrak{g}^*_{\mathbb{C}})$ can be regarded as a projectivized moment map for the action of the complex Lie group. This leads to a generalization of the hyperkähler reduction **8.7** for a quaternionic Kähler manifold M, found by Galicki [Ga], which we first formulate directly on M treating the 4-form Ω as if it were an ordinary symplectic 2-form.

Suppose, then, that X is a Killing vector field on a quaternionic Kähler manifold M. If $t\neq 0$, M is necessarily irreducible, and one may deduce that ∇X belongs to the holonomy algebra, so as in (5.4),

$$\nabla X \in \mathfrak{h} \subseteq \mathfrak{sp}(k)\oplus\mathfrak{sp}(1). \tag{9.14}$$

Since neither of the modules $\mathfrak{sp}(k)$, $\mathfrak{sp}(1)$ feature in Λ^4, the 3-form $X \lrcorner \Omega$ is closed. In the hyperkähler case $t=0$, (9.14) must be imposed as a hypothesis, but then

$$X \lrcorner \Omega = \sum_{i=1}^{3} d\mu^i \wedge \omega^i = d\Big(\sum_{i=1}^{3} \mu^i\omega^i\Big),$$

where μ^1,μ^2,μ^3 are the moment maps for the individual symplectic forms $\omega^1,\omega^2,\omega^3$.

9.7 Lemma *When $t\neq 0$, there exists a unique section ζ of the subbundle VM of Λ^2T^*M with fibre $\mathfrak{sp}(1)$ for which $d\zeta = X \lrcorner \Omega$.*

Proof. Define ζ to equal the $\mathfrak{sp}(1)$-component of ∇X. This will do the trick, at least up to a constant, for $d\zeta$ can be identified with

$$\nabla\zeta \in \Lambda^1 \otimes \mathfrak{sp}(1) \cong [\lambda_0^1\sigma^3] \oplus \Lambda^1,$$

and is completely determined by $\widetilde{R}(X)$, in the notation of (4.17). This tells us that the component of $\nabla\zeta$ in $[\lambda_0^1\sigma^3]$ vanishes, and the component in Λ^1 is proportional to tX. Uniqueness follows from the fact that if $\zeta' - \zeta$ is a non-zero parallel section of VM, then $\widetilde{R}(\zeta' - \zeta) = 0$, which is only possible if $t = 0$. □

More generally, the action of a group G of isometries preserving the $Sp(k)Sp(1)$ reduction of a quaternionic Kähler manifold M with $t \neq 0$ gives rise to a "moment section" $\mu \in \Gamma(M, VM)$. If $\mu^{-1}(0)$ denotes its zero set in M, the methods of chapter 8 can be extended to prove

9.8 Theorem [GL] *If the quotient $\mu^{-1}(0)/G$ is a manifold, then its induced metric is quaternionic Kähler.*

A basic example will illustrate the idea. The action of the circle group $U(1)$ on $\mathbb{H}^{k+1}$ corresponding to left multiplication by e^{it} induces an action on both $\mathbb{C}P^{2k+1}$ and $\mathbb{H}P^k$, and the subgroup of $Sp(k+1)$ of isometries that commute with $U(1)$ equals $U(k+1)$. It follows that the resulting quotient

$$\mu^{-1}(0)/U(1) = \{[q_0, \ldots, q_k] : \sum_{r=0}^{n} \overline{q}_r i q_r = 0\}/U(1) \tag{9.15}$$

is a complex Grassmannian, with twistor space a flag manifold F:

$$\begin{array}{ccc}
L^{-1} - i_0(F) & \hookleftarrow & \mathbb{H}^{k+1} - \{0\} \\
\downarrow \mathbb{C}^* & & \downarrow \mathbb{C}^* \\
F & \hookleftarrow & \mathbb{C}P^{2n+1} \\
\downarrow \mathbb{C}P^1 & & \downarrow \mathbb{C}P^1 \\
G_2(\mathbb{C}^{k+1}) & \hookleftarrow & \mathbb{H}P^k.
\end{array}$$

In fact, by reinterpreting the coordinates in (9.15), each horizontal arrow can be thought of as a sort of symplectic quotient. The relevance of the twistor space to this set-up was described in detail in [HKLR].

We have already seen that over the twistor space of any quaternionic Kähler manifold M with $t \neq 0$, the total space of the $\mathbb{C}^*$-bundle $L^{-1} - i_0(ZM)$ is complex symplectic. In fact, Swann has shown that this real $4(k+1)$-dimensional manifold has both a hyperkähler and a quaternionic Kähler metric (indefinite when $t < 0$), generalizing the obvious ones on $\mathbb{H}^{k+1}$ and $\mathbb{H}P^{k+1}$ [Sw]. In particular, this allows the theory of quaternionic Kähler manifolds to be subsumed into the theory of hyperkähler ones. When M is a Wolf space, $L^{-1} - i_0(Z)$ is a complex nilpotent coadjoint orbit (cf. (ii) at the end of the last chapter); in particular the hyperkähler metric for $M = G_2(\mathbb{C}^{k+1})$ is a degenerate limit of the Calabi metric on $\mu^{-1}(c)/U(1) \cong T^*\mathbb{C}P^k$.

Galicki and Lawson [GL] have described a modification of the above example giving rise to a host of quaternionic Kähler non-symmetric metrics at regular points of orbifolds. In particular, reducing to a 4-manifold, they discover infinitely many non-isometric self-dual Einstein metrics on weighted complex projective spaces. Their methods succeed in constructing smooth quotients when the scalar curvature t is negative.

A theorem of LeBrun [Le_2] associates to any real analytic conformal manifold N of signature $(3, k-1)$ a $4k$-dimensional quaternionic Kähler manifold M^{4k} with $t < 0$. The basic example is the non-compact dual M^8 of the Wolf space $\widetilde{Gr}_4(\mathbb{R}^8)$, each point x of whose twistor space Z defines a null line in the complex quadric $Q^4 \subset \mathbb{C}P^5$, which is a compactification and complexification of Minkowski space $\mathbb{R}^{3,1}$. In general, the twistor space ZM appears as an open set of the space of null geodesics of a complexification of N. The bundle $\widetilde{Gr}_3^+(TN)$ of 3-planes on which g is positive definite becomes identified with a hypersurface at infinity of M^{4k}.

Torsion and underlying structures

Let M be a $4k$-dimensional manifold with an $Sp(k)Sp(1)$-structure. From (9.3), one may deduce that $Sp(k)Sp(1)$ is a maximal Lie subgroup of $SO(4k)$ for $k > 1$. In contrast to the complex situation, there is no larger "symplectic" group preserving Ω, but despite this, we have seen some constructive analogies with symplectic geometry. The quaternionic Kähler condition is characterized by the vanishing of the $Sp(k)Sp(1)$-structure function $T_0(p) = (\nabla\Omega)(p)$ which, we recall from **2.2**, belongs to the space

$$\begin{aligned}\Lambda^1\otimes(\mathfrak{sp}(k)\oplus\mathfrak{sp}(1))^\perp &\cong [\lambda_0^1\sigma^1]\otimes[\lambda_0^2\sigma^2]\\ &\cong [\lambda_1^3\sigma^3]\oplus[\lambda_0^1\sigma^3]\oplus[\lambda_0^3\sigma^3]\oplus[\lambda_1^3\sigma^1]\oplus[\lambda_0^1\sigma^1]\oplus[\lambda_0^3\sigma^1]\end{aligned} \tag{9.16}$$

The last three summands are the components of $\Lambda^1\otimes[\lambda_0^2]$, so from (9.7), the $GL(k,\mathbb{H})\mathbb{H}^*$-structure function belongs to the three submodules of (9.16) involving σ^3. These three spaces therefore house the obstruction to the existence of a torsion-free $GL(k,\mathbb{H})\mathbb{H}^*$-connection. A manifold admitting a $GL(k,\mathbb{H})\mathbb{H}^*$-structure with a torsion-free connection is called *quaternionic*, and this is precisely the most general situation in which it is possible to define a complex manifold ZM satisfying **9.5**(i) [Bes, 14.68],[BE]. The torsion-free connection will not be unique, but can be made so by choosing a volume form for it to preserve. When $k=1$, the torsion condition is vacuous, since $GL(1,\mathbb{H})\mathbb{H}^*$ is identical to the pointwise conformal group $\mathbb{R}^+\times SO(4)$, but in this lowest-dimensional case, it is most appropriate to regard a quaternionic manifold as one with a self-dual conformal structure (see the remarks after **9.4**). As a higher-dimensional example, we cite the total space of the tangent bundle TM, where M is quaternionic Kähler, or 4-dimensional self-dual Einstein [S$_2$].

The existence of the twistor space explains why it is possible to define analogues of the Dolbeault $\overline{\partial}$-operator on an arbitary quaternionic manifold. Consider the space

$$\lambda_z^{r,0} = \textstyle\bigwedge^r(\lambda^1\otimes z) \cong \lambda^r\otimes z^r$$

of forms of type $(r,0)$ with respect to the complex line $z\in S^2$ as in (9.12). As z varies, $z^r = \underbrace{z\otimes\ldots\otimes z}_{r}$ generates the symmetric tensor product σ^r, and the $GL(k,\mathbb{H})\mathbb{H}^*$-module

$$\sum_{z\in S^2}[\![\lambda_z^{r,0}]\!] = [\lambda_0^r\sigma^r]$$

generalizes (7.9). If A^rM denotes the subbundle of $\bigwedge^r T^*M$ with fibre $[\lambda_0^r\sigma^r]$, we define a differential operator

$$D\colon A^rM\hookrightarrow \textstyle\bigwedge^r T^*M \xrightarrow{\ d\ } \bigwedge^{r+1}T^*M \xrightarrow{\ p\ } A^{r+1}M,$$

where p is the projection which is well defined by the action of $\mathbb{H}^*$. Note that $A^1M=T^*M$, so certainly $D^2\colon A^0M\to A^2M$ vanishes; when $k=1$, A^2M coincides with the bundle Λ^2_-M of (anti-)self-dual 2-forms. The next result can be deduced from (9.16).

9.9 Proposition *A manifold M with an $Sp(k)Sp(1)$-structure, $k \geqslant 2$, is quaternionic if and only if*

$$0 \to C^\infty(M) \longrightarrow \Gamma(T^*M) \xrightarrow{D} \Gamma(A^2M) \xrightarrow{D} \ldots \longrightarrow \Gamma(A^{2k}M) \to 0$$

is a complex.

There has been considerable investigation of the symmetries of the tensor $\nabla\Omega$ (e.g., in [FM]). When $k \geqslant 3$, all six components of (9.16) are to be found in the space Λ^5 in **9.2**, and the following result is proved by showing that the relevant homomorphism is injective.

9.10 Lemma [Sw] *A manifold of dimension $4k \geqslant 12$ admitting a closed 4-form Ω, linearly equivalent to* (9.4) *at each point is quaternionic Kähler.*

This result has particular relevance to the reduction procedure of **9.8**. However, it is in the eight-dimensional case, when $\Lambda^3 \cong \Lambda^5$, that (9.16) bears most similarity with (3.13). The following version of **3.5** begs an example of a compact 8-manifold satisfying the "symplectic" condition $d\Omega = 0$, but which is not quaternionic Kähler. Corresponding examples in the complex case are relatively easy to construct on certain compact nilmanifolds, formed by taking the quotient of a complex nilpotent Lie group by a discrete subgroup [CFG].

9.11 Figure Components of the $Sp(2)Sp(1)$-structure function $\nabla\Omega$

$[\lambda_1^3\sigma^3]$	$[\lambda_0^1\sigma^3]$
$[\lambda_1^3\sigma^1]$	$[\lambda_0^1\sigma^1]$

0	0
0	0

Quaternionic Kähler

0	0
?	?

Quaternionic

?	0
0	0

Symplectic

As quaternionic is to quaternionic Kähler, so hypercomplex is to hyperkähler. As indicated in the preceding chapter, a *hypercomplex* manifold M is one admitting a triple I_1, I_2, I_3 of complex structures with $I_1I_2 = -I_2I_1 = I_3$. A corollary of the next result is that the integrability of just two of these structures implies that $\sum_{i=1}^3 a^i I_i$ (for $\sum (a^i)^2 = 1$) is integrable.

9.12 Proposition [O] *Two anti-commuting almost complex structures on a manifold are both integrable if and only if the $GL(k,\mathbb{H})$-structure they determine admits a torsion-free connection. Any such connection is unique.*

Proof. Any torsion-free $GL(k,\mathbb{H})$-connection is automatically a torsion-free connection on each of the two $GL(2k,\mathbb{H})$-structures corresponding to I_1 and I_2. The latter are therefore complex structures. Uniqueness is a consequence of the injectivity of the fundamental sequence

$$\delta\colon (\mathbb{R}^{4k})^*\otimes\mathfrak{gl}(k,\mathbb{H}) \longrightarrow \Lambda^2(\mathbb{R}^{4k})^*\otimes\mathbb{R}^{4k}.$$

Existence of the connection follows from the fact that the $GL(k,\mathbb{H})$-structure function in $\operatorname{coker}\delta$ is determined by the Nijenhuis tensors of I_1 and I_2. Rather than verify this last assertion, we indicate an explicit proof when $k = 1$, and the bundles of $(1,0)$-forms of two anti-commuting almost complex structures I_1, I_2 can be expressed locally as

$$\mathrm{span}\{e^1 - ie^2, e^3 + ie^4\}, \quad \mathrm{span}\{e^1 - ie^3, e^4 + ie^2\}. \tag{9.17}$$

To say that a connection preserves the resulting $GL(1,\mathbb{H})$-structure means $\nabla I_1 = 0 = \nabla I_2$, or equivalently that the operator ∇_X leaves invariant the bundles (9.17), for any vector field X. If $\nabla e^r = \sum_{s,t} \alpha^r_{st} e^s \otimes e^t$, this is equivalent to the equations

$$\begin{cases} \alpha^1_{1t} = \alpha^2_{2t} = \alpha^3_{3t} = \alpha^4_{4t}, \\ \alpha^i_{jt} = -\alpha^j_{it}, \quad i \neq j, \\ \alpha^1_{2t} = \alpha^3_{4t},\ \alpha^1_{3t} = \alpha^4_{2t},\ \alpha^1_{4t} = \alpha^2_{3t}, \end{cases}$$

for all t. These equations are consistent with the zero torsion condition $de^r = \sum_{s<t}(\alpha^r_{st} - \alpha^r_{ts}) e^s \wedge e^t$, provided I_1, I_2 are integrable, which means that the bundles (9.17) are closed under exterior differentiation. The connection is then completely determined by formulae like

$$2\alpha^1_{12} = (\alpha^1_{12} - \alpha^1_{21}) + (\alpha^3_{41} - \alpha^3_{14}) - (\alpha^4_{24} - \alpha^4_{42}).$$

□

Given an oriented 3-dimensional manifold N, hypercomplex structures arise naturally from a study of the space of all parallelizations of N (the next remarks are based on an interpretation by Hitchin of work of Ashtekar et al. [AJS]). Let $e^1(t), e^2(t), e^3(t)$ be a triple of linearly independent 1-forms on a 3-dimensional manifold, which depend on a real parameter t. Then together with $e^4 = -dt$ on $N \times \mathbb{R}$, they define a hypercomplex structure as in (9.17) if the dual vector fields X_1, X_2, X_3 satisfy the evolution equations

$$\frac{dX_1}{dt} = [X_2, X_3], \quad \frac{dX_2}{dt} = [X_3, X_1], \quad \frac{dX_3}{dt} = [X_1, X_2].$$

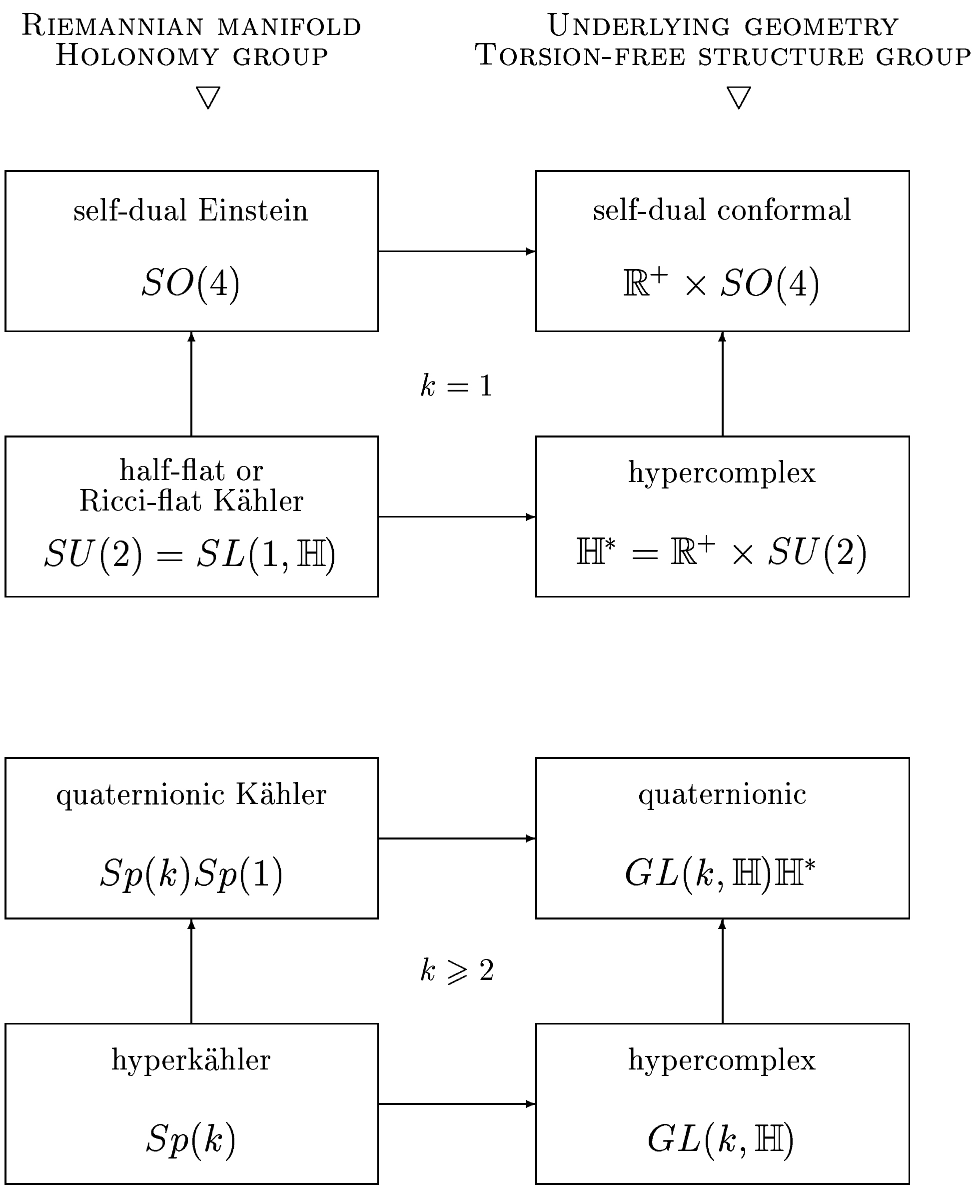
These equations are formally analogues to those of Nahm, in which the relevant Lie algebra is that of vector fields on N, and solutions correspond to critical points of the Chern-Simons invariant for connections on TN.

The last proposition is reminiscent of the "fundamental theorem of Riemannian geometry" **2.6**. The hypercomplex structure underlies a hyperkähler structure if and only if there exists a Riemannian metric which is covariant constant relative to the *Obata connection* determined by **9.12**; of course, the Obata and Levi Civita connections coincide in this case. As a corollary, observe that an irreducible hyperkähler metric is determined up to homothety by its associated complex structures.

The curvature of the Obata connection on a hypercomplex manifold M has two irreducible $GL(k,\mathbb{H})$-components, one of which is a skew-symmetric "Ricci tensor", which is simply the 2-form ω representing the induced curvature of the real line bundle $\bigwedge^{4k}T^*M$, and vanishes if and only the restricted holonomy reduces to $SL(k,\mathbb{H})$. The vanishing of both components implies integrability of the $GL(k,\mathbb{H})$ structure, and hence the existence of a flat affine structure on M; manifolds of this sort were considered by Sommese [So], and Kato [K]. A basic example of a compact integrable hypercomplex manifold which cannot admit any Kähler metric is the quaternionic Hopf surface $(\mathbb{H}^k - \{0\})/\Gamma \cong S^{4k-1} \times S^1$, where Γ is the group generated by the multiplicative action of a quaternion q with $|q| > 1$.

The definitions of hypercomplex and hyperkähler do not require modification when $k = 1$, but are closely linked by the special isomorphism $SL(1,\mathbb{H}) \cong SU(2)$. First observe that **9.12** implies that a 4-dimensional hypercomplex manifold M has a self-dual conformal structure; in addition, the 2-form $\omega \in \Gamma(M, \Lambda^2_+ M)$ is *self-dual*. Then ω vanishes identically if and only if M is locally hyperkähler; this leads to a classification of compact hypercomplex 4-manifolds [Boy_2].

9.13 Figure Quaternion-related structures



10 Classification Theorems

In this chapter, we address the question of which Lie groups arise as holonomy groups of irreducible Riemannian manifolds which are not symmetric. Symmetric spaces are characterized by the invariance of their curvature tensor R under parallel translation, so the departure from being symmetric is measured by the covariant derivative ∇R. We begin by examining properties not only of this tensor, but also higher derivatives of an Einstein curvature tensor.

An important characteristic of many symmetric spaces is the existence of subspaces of the tangent space of dimension greater than one, on which the curvature operator vanishes. The fact that such flat subspaces can exist only in symmetric situations is a deep result which was first proved by Berger as a corollary of his classification mentioned in our introduction. For completeness, we reproduce a direct proof of this fact of Simons, and derive Berger's list from the more well-known theory of Lie groups acting transitively on spheres.

Covariant derivatives of curvature

The reduction of the holonomy to a subgroup H of $SO(n)$ imposes conditions of varying severity on the curvature tensor R. There are also restrictions on the covariant derivative ∇R, which have their origin in the second Bianchi identity

10.1 Proposition $(\nabla_x R)_{yz} + (\nabla_y R)_{zx} + (\nabla_z R)_{xy} = 0.$

Proof. The curvature tensor arises from the Lie algebra valued 2-form $\Phi = (\theta \wedge \theta)R$ on the frame bundle LM, defined by (1.12). The structure equation $d\theta = -[\phi, \theta]$ implies that the horizontal component $(\theta \wedge \theta \wedge \theta)\nabla R$ of $d\Phi$ is zero. The proposition reinterprets this 3-fold anti-symmetrization. □

A more enlightening proof of this identity was given by Kazdan [Ka]; it appears as an integrability condition for the invariance of the definition of curvature under the action of the diffeomorphism group.

In the presence of a holonomy reduction, **10.1** implies that ∇R belongs to the kernel of the natural mapping

$$b_2 \colon \Lambda^1 \otimes \Re^H \longrightarrow \Lambda^3 \otimes \mathfrak{h}, \tag{10.1}$$

formed from the inclusion $\mathfrak{R}^H \subset \mathfrak{h}\otimes\mathfrak{h}$, and the composition

$$a_2\colon \Lambda^1\otimes\mathfrak{h} \hookrightarrow \Lambda^1\otimes\Lambda^2 \longrightarrow \Lambda^3. \tag{10.2}$$

In analogy to **4.2**, the reduced space of covariant derivatives of curvature tensors is therefore

$$\ker b_2 \;=\; \left(\Lambda^1\otimes\mathfrak{R}^H\right) \cap \left(\ker a_2 \otimes \mathfrak{h}\right).$$

It was shown by Berger [Be$_1$] that for all but a handful of subgroups H acting irreducibly on $O(n)$, the space $\ker b_2$ is zero. Notice that information may be inferred from the study of $\ker a_2$, which does not require knowledge of the space $\mathfrak{R}^H$ of curvature tensors.

A lot can also be said regarding the iterated covariant derivatives $\nabla^k R$, when the holonomy group is a proper subgroup of $SO(n)$. Rather than treat the latter case, we shall suppose that M is an Einstein manifold, which is automatic for many holonomy reductions. By Einstein, we understand that the component of R in the space Σ^2_0 vanishes, so that

$$\nabla R \;\in\; \Lambda^1\otimes(W \oplus \mathbb{R}) \subset \Lambda^1\otimes\mathfrak{R}.$$

The space W of Weyl tensors is not a summand of $\Lambda^1\otimes\Lambda^1$, so the tensor product $\Lambda^1\otimes W$ cannot contain Λ^1 as an $O(n)$-submodule. Since it is clear that the restriction of b_2 to the subspace $\Lambda^1\otimes\mathbb{R}$ containing the differential dt of the scalar curvature is non-zero, we have

$$\nabla R \in \Lambda^1\otimes W, \quad \text{and} \quad dt = 0, \tag{10.3}$$

whence the well-known fact that the scalar curvature of an Einstein manifold is constant.

10.2 Theorem *Let M be an Einstein manifold of dimension $n \geqslant 3$. Then $\nabla^k R$ is completely determined by its component in the irreducible $O(n)$-summand of $\bigotimes^k\Lambda^1\otimes W$ of highest weight, together with lower order derivatives $\nabla^\ell R$, where $0 \leqslant \ell \leqslant k-2$.*

Proof. For notational convenience, we shall suppose that $n = 7$, in order to apply **6.9**, although the general case requires no essential modification. We begin with the decompositions

$$\begin{aligned}\Lambda^1 \otimes W &= (1,0,0)\otimes(2,2,0)\\ &\cong (3,2,0)\oplus(2,2,1)\oplus(2,1,0),\\ \Lambda^3\otimes\Lambda^2 &= (1,1,0)\otimes(1,1,1)\\ &\cong (2,2,1)\oplus(2,1,0)\oplus(2,1,1)\oplus(1,1,1)\oplus(1,1,0)\oplus(1,0,0).\end{aligned}$$

Let R^k denote the component of $\nabla^k R$ in the highest weight summand

$$(k+2,2,0) \subset \Sigma^k \otimes W \subset \bigotimes^k \Lambda^1 \otimes W.$$

A quick check shows that the rank of b_2 is as large as Schur's lemma permits, so from (10.3),

$$\nabla R = R^1 \in (3,2,0) = \ker b_2.$$

The second derivative $\nabla^2 R$ belongs to $\Lambda^1 \otimes (3,2,0)$, and any of its components that do not lie in the kernel of the skewing map

$$\delta\colon \Lambda^1\otimes(3,2,0) \hookrightarrow \Lambda^1\otimes\Lambda^1\otimes W \longrightarrow \Lambda^2\otimes W$$

are determined by $\widetilde{R}(R)$ (cf. (4.17)). Using

$$\begin{aligned}\Lambda^1\otimes(3,2,0) &\cong (3,2,1)\oplus(3,1,0)\oplus(3,3,0)\oplus(2,2,0)\oplus(4,2,0),\\ \Lambda^2\otimes W &\cong (3,2,1)\oplus(3,1,0)\oplus(3,3,0)\oplus(2,2,0)\oplus(2,1,1)\oplus(1,1,0)\oplus(2,2,1),\end{aligned}$$

one checks that $\ker\delta = (4,2,0)$, so

$$\nabla^2 R = R^2 + \text{quadratic terms in } R.$$

The proof proceeds by induction; to prove the statement concerning R^{k+1} assuming those for R^{k-1} and R^k, the key points are

(i) R^k equals the component of $\nabla(R^{k-1})$ in $(k+2,2,0)$;

(ii) $(k+3,2,0) = \Big(\Lambda^1\otimes(k+2,2,0)\Big)\cap\Big(\Sigma^2\otimes(k+1,2,0)\Big)$;

(iii) all the components of ∇R^k, apart from R^{k+1} are determined by $\widetilde{R}(R^{k-1})$ and the covariant derivatives of components of ∇R^{k-1} orthogonal to R^k. □

As a simple application of **10.2**, an Einstein n-manifold M admitting a group of isometries whose isotropy subgroup at some point m acts as $SO(n)$ on T_mM must have constant curvature. This follows from the analyticity of Einstein metrics [DK], and the fact that the spaces containing R^k, $k \geqslant 0$, contain no invariants.

The covariant derivative ∇R of the curvature tensor of an Einstein 4-manifold has two irreducible components under the action of $SO(4)$, corresponding to the highest weight summands of $\Lambda^1 \otimes W_\pm$ [S$_2$]. Other information on covariant derivatives of the curvature tensor may be extracted from [Bou$_1$],[G$_4$].

Rank and Transitivity

Let H be a compact Lie group, regarded as a symmetric space as in (5.15), with the complement $\mathfrak{m}$ identified with a copy of the Lie algebra $\mathfrak{h}$ of H. Then the Lie algebra $\mathfrak{t}$ of a maximal torus can be thought of as a maximal abelian subalgebra of $\mathfrak{m}$. More generally, given a symmetric space G/H with

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$

a subalgebra $\mathfrak{a}$ of $\mathfrak{m}$ is necessarily abelian because $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$. Thus $\mathfrak{a}$ is *flat* in the sense that

$$[x, y] \;=\; R_{xy}$$

vanishes for all $x, y \in \mathfrak{a}$, and $\mathfrak{a}$ is tangent to a torus in G/H. Actually, any subspace $\mathfrak{b}$ for which

$$[[x, y], z] = R_{xy}z \;\in\; \mathfrak{b}, \quad x, y, z \in \mathfrak{b}$$

defines a symmetric algebra $[\mathfrak{b}, \mathfrak{b}] \oplus \mathfrak{b}$, and a totally geodesic submanifold of G/H. The subspace $\mathfrak{b}$ is known as a Lie triple system [He].

Any two maximal abelian subalgebras of $\mathfrak{g}$ are $\mathrm{Ad}\,G$-conjugate; likewise any two maximal abelian subspaces of $\mathfrak{m}$ are $\mathrm{Ad}\,H$-conjugate, and their common dimension is called the *rank* of G/H. Given such a subspace $\mathfrak{a}$ of $\mathfrak{m}$, the analogy just mentioned suggests studying the action of $\mathrm{ad}(\mathfrak{a})$ on $\mathfrak{g}$. In particular, any $u, v \in \mathfrak{a}$ define an endomorphism

$$\begin{aligned} T(u,v)x &= -\mathrm{ad}(v)\mathrm{ad}(u)x \\ &= [[u,x],v] \qquad\qquad (10.4) \\ &= R_{ux}v \end{aligned}$$

of $\mathfrak{m}$. The first Bianchi identity **4.1** implies that $R_{ux}v = R_{vx}u$, so that $T(u,v) = T(v,u)$. Moreover,

10.3 Lemma *$\{T(u,v) : u, v \in \mathfrak{a}\}$ is a set of commuting symmetric endomorphisms of $\mathbb{R}^n$.*

Proof. The fact that $g(T(u,v)x,y) = g(x,T(u,v)y)$ is a consequence of the symmetry $g(R_{uv}x,y) = g(R_{xy}u,v)$. From (4.1) and the invariance of the curvature tensor R, we have

$$R_{Ax,y} + R_{x,Ay} = 0, \quad A \in \mathfrak{h}. \tag{10.5}$$

The fact that $T(u,v)\circ T(x,y) = T(x,y)\circ T(u,v)$ follows by taking $A = R_{uv}$. □

There exists an orthonormal basis $\mathbb{R}^n$ consisting of simultaneous eigenvectors for the $T(u,v)$'s. One such eigenvector e satisfies

$$T(u,v)e = k(u,v)e,$$

k being the corresponding symmetric bilinear form on $\mathfrak{a}$. If $k \neq 0$, then $R_{we}w = k(w,w)e \neq 0$ for some $w \in \mathfrak{a}$, and

$$\begin{aligned} R_{ue} &= \frac{1}{k(w,w)} R_{u,\,R_{we}w} \\ &= \frac{-1}{k(w,w)} R_{R_{we}u,\,w} \\ &= \frac{k(w,u)}{k(w,w)} R_{we}. \end{aligned} \tag{10.6}$$

Applying both sides to the vector v, one sees that the rank of the bilinear form k is equal to one, and determines a "root plane"

$$U_k = \{u \in \mathfrak{a} : R_{ue} = 0\}$$

of codimension one in $\mathfrak{a}$.

As an example, consider the Grassmannian

$$\widetilde{Gr}_p(\mathbb{R}^{p+q}) = \frac{SO(p+q)}{SO(p)\times SO(q)}$$

of oriented real p-dimensional subspaces of $\mathbb{R}^{p+q}$, with $p \leqslant q$. Its isotropy representaion may be identified with the tensor product $\mathbb{R}^p \otimes \mathbb{R}^q$, whose factors have respective orthonormal bases $\{x_1,\ldots,x_p\}$, $\{y_1,\ldots,y_q\}$. A maximal abelian subalgebra of $\mathfrak{m}$ is given by

$$\mathfrak{a} = \operatorname{span}\{x_i \otimes y_i : 1 \leqslant i \leqslant p\}, \tag{10.7}$$

and the orthogonal complement of any $x_i \otimes y_i$ in $\mathfrak{a}$ is a possible U_k. When $p = 2$, the description (3.17) gives a very clear interpretation of (10.7); it is tangent to a torus $S^1 \times S^1$ lying in some totally geodesic submanifold $\mathbb{C}P^1 \times \mathbb{C}P^1 = Q^2$.

Observe that the product $M_1 \times M_2$ of two symmetric spaces always has rank greater than one.

10.4 Lemma *An irreducible n-dimensional symmetric space has rank one if and only if its holonomy group H acts transitively on the sphere $S^{n-1} \subset \mathfrak{m}$.*

Proof. The action of H on $x \in S^{n-1}$ gives rise to a map $\ell_x: H \to S^{n-1}$ with differential

$$d\ell_x: \mathfrak{h} = T_e H \longrightarrow T_x S^{n-1} = \{x\}^\perp.$$

It is not hard to see that $d\ell_x$ is surjective for all $x \in S^{n-1}$ if and only if H is transitive. Since the Lie algebra $\mathfrak{h}$ is generated by curvature operators, surjectivity fails precisely when there exists $y \in \{x\}^\perp$ for which

$$0 = g(R_{uv}x, y) = g(R_{xy}u, v) \tag{10.8}$$

for all $u, v \in \mathbb{R}^n$. The span of x and y is then an abelian subspace of $\mathfrak{m}$. □

Transitivity on the sphere may be thought of as an infinitesimal consequence of the fact that any rank one symmetric space is *two-point homogeneous*, meaning that any two pairs of equidistant points can be interchanged by a suitable isometry. Closely connected with this is the fact that rank one symmetric spaces have all their geodesics simple closed and of equal length [Ca$_4$]. It follows from **5.2** that an irreducible symmetric spaces has rank one if and only if its sectional curvature is non-zero for every 2-plane. The list

$$S^n, \quad \mathbb{C}P^m, \quad \mathbb{H}P^k, \quad \mathbf{O}P^2 = \frac{F_4}{Spin(9)}$$

of simply-connected ones of compact type can be deduced from the following classification, a special case of the theorem of Montgomery and Samuelson [MS], [B$_1$].

10.5 Theorem *Let H be a closed subgroup of $O(n)$ acting acting transitively and effectively on the sphere S^{n-1}. Then H is one of*

$$SO(n), \quad U(\tfrac{n}{2}), \quad SU(\tfrac{n}{2}),$$
$$Sp(\tfrac{n}{4}), \quad Sp(\tfrac{n}{4})U(1), \quad Sp(\tfrac{n}{4})Sp(1),$$
$$G_2 \ (n=7), \quad Spin(7) \ (n=8), \quad Spin(9) \ (n=16).$$

Proof. Since H is compact, its Lie algebra has the form $\mathfrak{h} = \mathfrak{z} \oplus [\mathfrak{h},\mathfrak{h}]$, where $\mathfrak{z}$ is its centre, and $[\mathfrak{h},\mathfrak{h}]$ is a sum of simple ideals. We first show that $\mathfrak{h}$ is in fact "nearly simple", in the sense that it is the direct sum of a simple ideal and at worst an ideal isomorphic to $\mathfrak{sp}(1)$ or $\mathfrak{u}(1)$. Suppose first that $\mathfrak{h}$ equals the direct sum $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ of two non-zero ideals. Certainly $\mathfrak{h}$ acts irreducibly on $\mathbb{R}^n$, which is therefore isomorphic to one of

(i) $[V_1] \otimes [V_2]$, and each V_i has an antilinear ε with $\varepsilon^2 = +\mathbf{1}$;

(ii) $[V_1 \otimes V_2]$, and each V_i has an antilinear ε with $\varepsilon^2 = -\mathbf{1}$;

(iii) $[\![V_1 \otimes V_2]\!]$.

The three cases correspond to the imposition of real, quaternionic or complex structures respectively; in each one, V_i is a complex irreducible $\mathfrak{h}_i$-module.

If $x \in \mathbb{R}^n$, we know that $\mathfrak{h}(x) = \mathfrak{h}^\perp$. In case (i) the example (10.7) shows that one V_i must be 1-dimensional, which implies that $\mathfrak{h}_i = 0$. In case (ii), any real vector x can be written $v_1 \otimes v_2 + \varepsilon v_1 \otimes \varepsilon v_2$, and

$$\mathfrak{h}(x) \;=\; \mathfrak{h}v_1 \otimes v_2 + v_1 \otimes \mathfrak{h}v_2 + \mathfrak{h}(\varepsilon v_1) \otimes \varepsilon v_2 + \varepsilon v_1 \otimes \mathfrak{h}(\epsilon v_2).$$

Hence one V_i is generated by v_i and εv_i, which means that $\mathfrak{h}_i$ acts as $\mathfrak{sp}(1)$. Similarly, in (iii), one $\mathfrak{h}_i$ must be isomorphic to $\mathfrak{u}(1)$.

It is now possible to deduce the theorem from the classification of simple Lie algebras and their representations. Assuming $\mathfrak{h}$ has rank greater than one, $\mathbb{R}^n$ cannot equal the adjoint representation of $\mathfrak{h}$, for any two vectors x, y in the Lie algebra $\mathfrak{t}$ of a maximal torus will satisfy (10.8). For $n \geqslant 4$, the isotropy subgroup is necessarily non-trivial for $n \geqslant 4$, and one need check only real irreducible representations of dimension less than or equal to $\dim \mathfrak{h}$. Experience in the methods of chapter 6 shows that all the representations for which $\mathfrak{h}(x) = \{x\}^\perp$ occur in the following list.

$\mathfrak{su}(m),\ m \geqslant 2$ $\mathfrak{u}(m)$	$[\![\lambda^{1,0}]\!] \cong \mathbb{R}^{2m}$
$\mathfrak{sp}(k),\ k \geqslant 1$ $\mathfrak{sp}(k) \oplus \mathfrak{sp}(1)$	$[\lambda^1 \otimes \mathbb{C}^2] \cong \mathbb{R}^{4k}$
$\mathfrak{so}(n),\ n \geqslant 7$	$\Lambda^1 \cong \mathbb{R}^n$ $\Delta \cong \mathbb{R}^8\ (n=7),\ \Delta \cong \mathbb{R}^{16}\ (n=9)$
$\mathfrak{g}_2$	$\mu \cong \mathbb{R}^7$

The basic representation μ of G_2, and the spin representation Δ of $Spin(2m-1)$ are discussed in the last two chapters. $\square$

The homogeneous spaces

$$\frac{U(m)}{U(m-1)}, \quad \frac{Sp(k)Sp(1)}{Sp(k-1)Sp(1)}, \quad \frac{Spin(9)}{Spin(7)} \tag{10.9}$$

occur naturally as the *distance spheres* in $\mathbb{C}P^m$, $\mathbb{H}P^k$ and $\mathbf{O}P^2$ respectively, and in the last two cases the induced metrics include non-standard Einstein ones [J],[BK]. In fact, S^{15} possesses exactly three non-homothetic homogeneous Einstein Riemannian metrics [Z_1]. The spaces (10.9) all have a reducible isotropy representation that reflects their realization as sphere bundles over $\mathbb{C}P^{m-1}$, $\mathbb{H}P^{k-1}$ and S^8 respectively; the $\mathbb{H}P^1$ case is related to a construction in chapter 12.

Results of Berger and Simons

As usual, we fix an orthonormal frame p in order to identify the tangent space T_mM with $\mathbb{R}^n$, and the holonomy group $H = H(p)$ with a subgroup of $O(n)$. In order to apply the above ideas to an n-dimensional Riemannian manifold M, it is necessary to compensate for the fact that the curvature tensor $R = R(p)$ will not be invariant by the holonomy group. It then becomes appropriate to consider the *orbit* of R in the space $\mathfrak{R}^H \subset \Lambda^2 \otimes \mathfrak{h}$.

10.6 Theorem *If M is irreducible, then either the holonomy group H acts transitively on S^{n-1} or its identity component H^0 acts trivially on the space of curvature tensors $\Re^H$.*

Proof. We follow [Si], to which the reader should refer for more details. To prove the theorem, we may suppose that $H = H^0$ is a closed connected subgroup of $O(n)$ acting irreducibly on $\mathbb{R}^n$, but not transitively on S^{n-1}, and that $J \subset \Re^H$ is an H-orbit. We shall eventually conclude that J consists of one point, irrespective of the common dimension d for $R \in J$ of the ideal

$$\mathfrak{h}^R = \operatorname{span}\{h^{-1} \circ R_{xy} \circ h : h \in H\},$$

of $\mathfrak{h}$ (cf. (2.11)). First, though, consider the more modest hypothesis that the conclusion is valid whenever (n, d) is replaced by $(n', d') \prec (n, d)$, where $\prec$ denotes the lexicographic ordering. The case $n = 2$ is trivial.

The assumption that $H \subset O(n)$ does not act transitively on the sphere implies that there exist linearly independent vectors $x, y \in \mathbb{R}^n$ such that $R_{xy} = 0$ for all $R \in J$. This follows by simply replacing (10.8) by

$$0 = g(h^{-1} R_{uv} hx, y) = g(R_{hx,hy} u, v)$$

in the proof of lemma **10.4**. Let W be a *maximal* subspace $\mathbb{R}^n$ for which $R_{xy} = 0$ for all $x, y \in W$ and $R \in J$. Given $R \in J$, a symmetric bilinear form $T_R(u, v)$ is defined on W as in (10.4), and provided $J \neq \{0\}$, we may always choose W so that the simultaneous eigenvectors are not all trivial ones. Given an eigenvector e with eigenvalue $k \neq 0$, the hyperplane $U_{R,k} = \ker k$ of W is used to define

$$M_{R,k} = \{v \in \mathbb{R}^n : S_{uv} = 0,\ \forall u \in U_{R,k},\ S \in J\}.$$

The generalization

$$S_{ue} = \frac{k(w, u)}{k(w, w)} S_{we}, \quad \forall u \in W,\ S \in J$$

of (10.6) implies that $M_{R,k}$ contains $W \oplus \mathbb{R}e$. If $M_{R,k}$ and $M_{R',k'}$ are distinct, then

$$M_{R,k} \cap M_{R,k'} = W. \tag{10.10}$$

For let x belong to the intersection; then $S_{wx} = 0$ for all $w \in W = U_{R,k} + U_{R',k'}$, and so $x \in W$ by maximality. We also have

$$S_{Au,m} = -S_{u,Am}, \quad \forall u \in U_{R,k},\ m \in M_{R,k},\ S \in J,\ A \in \mathfrak{h}, \tag{10.11}$$

which implies, for each $R \in J$, that

$$R_{xy}(M_{R,k}) \subseteq M_{R,k}, \quad \forall x, y \in M_{R,k}. \tag{10.12}$$

This property is expressed by saying that $M_{R,k}$ is *totally geodesic.*

A central assertion in the proof is that

$$\mathbb{R}^n = \sum M_{R,k}, \tag{10.13}$$

where the sum ranges over all $R \in J$, and over all non-zero eigenvalues k arising for each R. Suppose that y is orthogonal to every $M_{R,k}$. By the maximality of W, to prove (10.13), it suffices to show that $A = R_{wy}$ vanishes for all $w \in W$ and $R \in J$. For each R, y is a sum of trivial eigenvectors, so it belongs to

$$Z = \{z : T_R(u,v)z = 0, \ \forall u, v \in W, \ R \in J\}.$$

This is a proper subspace of $\mathbb{R}^n$ (use the notion of sectional curvature (4.6)), and is totally geodesic as in (10.12) [Si, lemma 7]. This means that the restriction $\hat{R} = R|_Z$ belongs to the space $\Re^{\hat{H}}$ of curvature tensors on Z with values in the Lie algebra of

$$\hat{H} = \{h|_Z : h \in H, \ h(Z) \subseteq Z\}.$$

If $\hat{H}$ acts irreducibly on Z, our inductive hypothesis implies that either $\hat{R}$ is the curvature tensor of a symmetric space or $\hat{R}$ acts transitively on the sphere in Z. In the former case, using (5.3), $\hat{R}_{wy}w = 0$ implies

$$0 = \hat{R}_{wy}z = Az, \quad \forall z \in Z. \tag{10.14}$$

In the latter case, (10.14) also follows as a consequence of the fact that $\hat{R}_{hw,z}hw$ vanishes for all $z \in Z$ and $h \in \hat{H}$. In general, (10.14) may be established by decomposing Z into irreducible summands. To complete the proof of (10.13), it remains to show that $Am = 0$ whenever m belongs to some $M_{R,k}$. Now, if $u \in U_{R,k}$, then $Au = 0$ and from (10.11), $Am \in M_{R,k}$; but (10.12) implies that $0 = g(R_{m,Am}w, y) = g(Am, Am)$.

A corollary of (10.13) is the existence of at least two distinct spaces as in (10.10). For if not, $\mathbb{R}^n = M_{R,k}$, and it is easy to see with the irreducibility assumption that this implies that $J = \{0\}$. We now replace our choice of original choice of W by another subspace hW for some $h \in H$. This does not affect the validity of the above arguments, and in view of (10.10), it is possible to choose a basis $\{x_i\}$ of $\mathbb{R}^n$, each element of which belongs to either hW or some $M^{\perp}_{R,hW,k}$. If $A = R_{hw,x_i}$ for some $R \in J$ and $w \in W$, then $\mathfrak{h}^{AR}$ is a *proper* ideal of $\mathfrak{h}^R$. This follows from the fact that if e is an eigenvector of $T_R(u,v)$ with non-zero eigenvalue k, then $S_{hw,e} = 0$ whenever S belongs to the orbit of H containing AR [Si, lemma 11].

Because H acts on $\mathbb{R}^n$ irreducibly, h can be varied in the above argument so as to manufacture a basis $\{A_i\}$ of $\mathfrak{h}^R$ such that $\dim \mathfrak{h}^{A_i R} < d$ for all i. This can be extended to a basis of $\mathfrak{h}$ with the same property, since any complementary ideal to $\mathfrak{h}^R$ in $\mathfrak{h}$ annihilates R. By the inductive hypothesis, the resulting elements $A_i R$ must all be annihilated by $\mathfrak{h}$, and it follows that $0 = A_i^2(R) = -A_i^t A_i(R)$, whence $A_i R = 0$ and R is H-invariant. With the value n fixed, the inductive hypothesis can then be verified for all d. □

We can conclude from **10.6** that if the holonomy group H of a Riemannian manifold M is not transitive on S^{n-1}, then at each point q in the holonomy bundle $Q = Q(p)$, the tensor $R(q)$ is annihilated by the holonomy algebra $\mathfrak{h}$. Simons's ingenious proof relied solely on the *first* Bianchi identity, but we need to resort to the second Bianchi identity to assert that $R(p)$ is actually constant on Q. From the arguments that led up to **5.2**, we do know that $\mathfrak{h} = \mathfrak{h}^{R(q)}$ is generated by $R(q) = c(q)(K^{\mathfrak{h}} + \frac{1}{n-2}K^{\mathfrak{so}(n)})$. But then $c(q)$ is a universal constant times the scalar curvature, which is itself constant by virtue of (10.1). Therefore R itself is constant on the holonomy bundle, or in symbols, $\nabla R = 0$.

Theorems **10.5**, **10.6** lead to Berger's theorem [Be_1] cited in the introduction. Combined with **5.6**, it gives more or less the complete picture concerning the possible holonomy groups of a simply-connected irreducible Riemannian manifold, although we may tie up a few loose ends. As Berger observed, the existence of $Sp(k)U(1)$ as a holonomy group is precluded by the equality

$$\mathfrak{R}^{Sp(k)U(1)} = \mathfrak{R}^{Sp(k)},$$

which arises from an easy adaptation of the proof of **9.3**. Later, it was shown by Alekseevskii [Al_1], and Brown and Gray [BG] that $Spin(9)$ is eliminated from the list of non-symmetric holonomy groups on similar grounds. In fact, $\mathfrak{R}^{Spin(9)}$ is generated by the curvature tensor of the Cayley projective plane $\mathbf{O}P^2$ (see (12.12)).

10.7 Table Holonomy groups of irreducible Riemannian manifolds

HOLONOMY GROUPS OF SYMMETRIC SPACES

(i) H a maximal compact subgroup of a centreless non-compact group G, with $G_{\mathbb{C}}$ simple, $n = \dim G - \dim H$; e.g. H one of:

(ii) H a simple centreless compact group, $n = \dim H$

$SO(n)$

$U(\frac{n}{2})$

$Sp(\frac{n}{4})Sp(1)$

$SU(\frac{n}{2})$

$Sp(\frac{n}{4})$

G_2, $n = 7$

$Spin(7)$, $n = 8$

HOLONOMY GROUPS OF METRICS NOT LOCALLY SYMMETRIC

The lists of Cartan and Berger combine to give the above table of holonomy groups which, as we shall soon see, are all realized by complete Riemannian metrics. The corresponding problem in the non-simply-connected case is non-trivial; for symmetric spaces it leads to the analogue of the space form problem [W$_3$]. Since the groups G_2, $Spin(7)$, $U(m)$, $Sp(k)Sp(1)$ are their own normalizers, one need only worry about the Ricci-flat Kähler case. See also the remarks concerning flat manifolds in chapter 2. Another situation which is not entirely straightforward is that in which M is a non-complete reducible Riemannian manifold; although **10.5** can be applied to each factor H_i of **2.9**, it remains to relate the behaviour of the metric at different points on M.

The problem of reducibility is much more serious for a metric with indefinite signature, which has been studied by Wu [Wu],[Bes]. However, Berger's classification of groups acting irreducibly encompassed the indefinite case. Apart from some obvious non-compact forms of the groups above with n, m or k replaced by (p, q), and more exceptional cases, the list includes $SO(m, \mathbb{C})$ (regarded as a subgroup of $SO_0(m, m)$, cf. (5.16)) and the potential holonomy group $SO(k, \mathbb{H})$ (regarded as a subgroup of $SO_0(2k, 2k)$, cf. (5.14)) [Be$_1$],[Br$_2$].

A final application illustrates how **10.8** can be combined with other major theorems of Riemannian geometry.

10.8 Theorem [Bea$_1$,Bo$_1$,FW] *Let M be a compact Kähler manifold with zero Ricci tensor. Then some finite covering of M is isomorphic to a product*

$$T \times M_1 \times \cdots \times M_r \times N_1 \times \cdots \times N_s,$$

where T is a complex torus, and each M_i (respectively N_j) is a compact simply-connected manifold with holonomy group equal to $SU(m_i)$, $m_i \geqslant 3$ (respectively $Sp(k_j)$, $k_j \geqslant 1$). In particular, the holonomy group H of M is compact.

Proof. The decomposition theorem of de Rham's implies that the universal covering $\widetilde{M}$ of M is isometric to the product of $\mathbb{C}^p$ and various simply-connected irreducible factors, each of these will also have Ricci-flat Kähler metrics. If one of these irreducible factors is not compact, it must contain a *line*, that is a geodesic defined for all real values of its parameter, minimizing the distance between any two of its points. The Cheeger-Gromoll theorem (valid whenever the Ricci tensor is non-negative) [CG],[EsH] would then imply that the line splits off as a factor, which is a contradiction. By **5.2**, **10.7**, we may now write

$$\widetilde{M} \cong \mathbb{C}^p \times M_1 \times \cdots \times M_r \times N_1 \times \cdots \times N_s,$$

with M_i, N_j as in the theorem.

Because $M_1 \times \cdots \times N_s$ admits no covariant constant (and by Bochner, no holomorphic) vector fields, its group of isometric automorphisms is finite. It follows that if Γ is the subgroup of $\pi_1(M)$ consisting of elements that act trivially on each M_i and N_j, then $\mathbb{C}^p/\Gamma$ is a compact manifold. If Γ' is the subgroup of Γ of pure translations, the proof is completed by setting $T = \mathbb{C}^p/\Gamma'$, which is a torus by the remarks surrounding (2.10)). $\square$

Berger observed the following corollary of the classification theorem. The existence on an n-dimensional irreducible Riemannian manifold of a non-zero *covariant constant* exterior k-form, $0 < k < n$, implies that M is Einstein, or that the form is a multiple of a Kähler 2-form. This applies in particular to our fundamental 4-form **5.3**. In view of **9.11** and **12.4**, it would be interesting to enumerate the circumstances in which a *closed* 4-form guarantees the existence of an Einstein metric.

11 The Holonomy Group $\mathbf{G_2}$

The aim of this chapter is to establish the existence of Riemannian metrics whose holonomy group equals the exceptional 14-dimensional Lie group G_2, regarded as a subgroup of $SO(7)$. After definitions based on the existence of a subgroup $SO(4)$, we discuss G_2-structures on seven-dimensional manifolds, in much the same spirit as the geometries arising from $U(m)$ and $Sp(k)Sp(1)$ were tackled previously. The relevant structure function can be read off from the exterior derivatives of a 3-form and a 4-form, both of which therefore play a crucial role in the theory.

Of course, obtaining metrics with holonomy group strictly contained in G_2 is an easy matter, an extreme example being the flat metric on $\mathbb{R}^7$. However, even this case should not be dismissed, because it is important for an understanding of exactly how the G_2-structure on $\mathbb{R}^7$ is built out of a certain structure on the sphere S^6. An attempt to duplicate this situation by replacing S^6 with the twistor space of a 4-manifold leads us to the required metrics with exceptional holonomy.

Relations with SO(4)

A convenient way of describing any of the exceptional Lie groups is to choose a suitable subgroup from which to build up a picture of the larger group. This approach works most effectively when the resulting coset space is symmetric. The list of symmetric quaternionic Kähler manifolds includes the 8-dimensional coset space $G_2/SO(4)$, and in order to identify its isotropy representation, we first undertake a momentary digression on representations of $SO(4)$.

In chapter 7, we managed quite well with the description of $SO(4)$ as the double-covering of $SO(3)\times SO(3)$, but greater insight is obtained by treating $SO(4) = Sp(1)Sp(1)$ as a special case of (9.1). From this point of view, any irreducible representation of $SO(4)$ is real and has the form

$$\Delta^{p,q} = [\sigma_+^p \otimes \sigma_-^q], \quad p+q \text{ even}, \tag{11.1}$$

where σ_+^p, σ_-^q are the symmetric powers of the basic representations of the two $Sp(1)$-factors. For reasons of dimension and irreducibility, $SO(4)$ must act on the tangent space of $G_2/SO(4)$ as follows. If we decree that $\sigma = \sigma_+^1$, then the other factor λ^1 in

(9.2) equals σ_-^3, so as to define a non-trivial inclusion of the second $Sp(1)$ factor in $Sp(2)$. This leads to the symmetric Lie algebra

$$\begin{aligned} \mathfrak{g}_2 &= \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus [\sigma_+^1 \otimes \sigma_-^3] \\ &\cong \Delta^{2,0} \oplus \Delta^{0,2} \oplus \Delta^{1,3}, \end{aligned} \tag{11.2}$$

in which the bracket can be made explicit using the appropriate $SO(4)$-equivariant homomorphisms.

We define G_2 to be the simply-connected Lie group with the above Lie algebra, but it is more useful to seek a low-dimensional representation of $SO(4)$ whose Lie algebra of endomorphisms contains the right-hand side of (11.2). For this purpose, consider the 7-dimensional space

$$\begin{aligned} \mu &= \Delta^{1,1} \oplus \Delta^{0,2} \\ &= \Lambda^1 \oplus \Lambda^2_-, \end{aligned} \tag{11.3}$$

where the second line refers to standard representations of $SO(4)$ discussed in chapter 7. In terms of (7.4) and (7.5), μ has an oriented orthonormal basis

$$\begin{aligned} &E^1 = dx^1, \quad E^2 = dx^2, \quad E^3 = dx^3, \quad E^4 = dx^4, \\ &E^5 = \omega^1, \quad E^6 = \omega^2, \quad E^7 = \omega^3, \end{aligned} \tag{11.4}$$

that is compatible with the inclusion $SO(4) \subset SO(7)$. This basis enables us to identify $\bigwedge^2 \mu$ with $\mathfrak{so}(7)$. We resume the habit, begun in chpater 7, of omitting the exterior product symbol $\wedge$.

11.1 Lemma *G_2 is the subgroup of $GL(7,\mathbb{R})$ that preserve the 3-form*

$$\varphi = (E^1E^2 - E^3E^4)E^5 + (E^1E^3 - E^4E^2)E^6 + (E^1E^4 - E^2E^3)E^7 + E^5E^6E^7,$$

and lies in $SO(7)$.

Proof. This result is analogous to **9.1**. As a temporary measure, let $\mathfrak{h}$ denote the subalgebra of $\operatorname{End}\mu$ annihilating φ. By construction, φ is $SO(4)$-invariant, so $\mathfrak{h}$ contains $\mathfrak{so}(4)$. In terms of $SO(4)$-modules, $\operatorname{End}\mu \cong \mathbb{R} \oplus \bigodot^2_0 \mu \oplus \bigwedge^2 \mu$, where

$$\begin{aligned} \textstyle\bigodot^2_0 \mu &\cong \mathbb{R} \oplus \Delta^{1,1} \oplus \Delta^{2,2} \oplus \Delta^{0,4} \oplus \Delta^{1,3} \\ \textstyle\bigwedge^2 \mu &\cong \mathfrak{so}(4) \oplus \Delta^{0,2} \oplus \Delta^{1,1} \oplus \Delta^{1,3}. \end{aligned} \tag{11.5}$$

The element $E^1 \wedge E^5 + E^4 \wedge E^6 \in \bigwedge^2 \mu$ defines an endomorphism

$$(E^1, E^2, E^3, E^4, E^5, E^6, E^7) \mapsto (E^5, 0, 0, E^6, -E^1, -E^4, 0)$$

that does not preserve the decomposition (11.3), but nevertheless annihilates φ. It belongs to the subspace $\Delta^{1,3} \cong \Lambda^1 \otimes \Lambda^2_-$ which by Schur's lemma must therefore lie in $\mathfrak{h}$. A quick check of elements in other irreducible components in (11.5) confirms that $\mathfrak{h}$ is the subalgebra of $\mathfrak{so}(7)$ isomorphic to $\mathfrak{so}(4) \oplus \Delta^{1,3}$, which can be identified with $\mathfrak{g}_2$. An explanation of the fact that the stabilizer of φ is simply-connected can be found in [Br$_2$]. □

From **11.1**, $\mathfrak{g}_2$ is the kernel of the mapping

$$\operatorname{End} \mu \cong \mu \otimes \mu \longrightarrow \bigwedge^3 \mu, \tag{11.6}$$

defined by $\alpha \mapsto \alpha(\varphi)$. Since $\dim(\bigwedge^3 \mu) = 35 = \dim(\operatorname{End} \mu) - \dim(\mathfrak{g}_2)$, this mapping is surjective, and the orbit $GL(7, \mathbb{R})/G_2$ of φ is open in $\bigwedge^3 \mu$. The only other open orbit is generated by the form φ^* obtained from φ by changing the coefficient of $E^5 E^6 E^7$ to -1. The Lie algebra of its stabilizer G_2^* is the symmetric Lie algebra dual to (11.2); this is a subalgebra of $\mathfrak{so}(4,3)$, and has a non-zero projection to both the components $\Delta^{1,3}$ that appear in (11.5). Thus any 3-form that is "positive" and "non-degenerate" in an appropriate sense has stabilizer G_2. The reader is invited to work out explicitly how the 3-form determines an underlying positive definite metric g on μ.

There are many essentially equivalent ways of setting up the algebra needed to understand the action of G_2 on $\mathbb{R}^7$. The decomposition (11.3) corresponds to the description

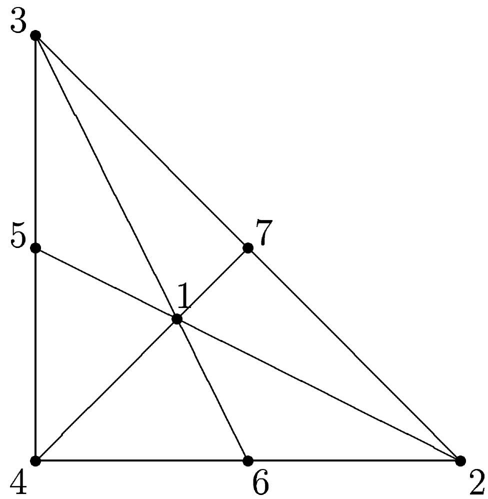
$$\begin{aligned} \mathbf{O} &= \operatorname{Re} \mathbf{O} \oplus \operatorname{Im} \mathbf{O} \\ &= \mathbb{R} \oplus \operatorname{Im} \mathbb{H} \oplus \mathbb{H} \end{aligned}$$

of the octonians or Cayley numbers in terms of quaternions. The representation μ plays the role of the space $\operatorname{Im} \mathbf{O}$ of imaginary octonians, and Cayley multiplication

$$\begin{array}{ccc} \bigwedge^2 \mu & \longrightarrow & \mu \\ X \wedge Y & \mapsto & X \times Y, \end{array} \tag{11.7}$$

is defined by setting $g(x \times y, z) = \varphi(x, y, z)$. The fact that (11.7) is an *isometry* puts it on a par with the ordinary vector cross product of $\mathbb{R}^3$. With our choice of basis elements, φ and therefore (11.7) are encoded in the well-known configuration below (referred to, for example, in [Fr]).

11.2 Figure Fano projective plane with orientations



A subspace of μ the form span$\{x, y, x \times y\}$ is called *associative*, and corresponds to the imaginary part of a quaternionic subalgebra of $\mathbf{O}$, or equivalently the subspace Λ^2_- in (11.3). The set of all associative subspaces is parametrized by $G_2/SO(4)$. If we map a 3-plane with an oriented orthonormal basis $\{x, y, z\}$ to the element $x \wedge y \wedge z$ in $\bigwedge^3 \mathbb{R}^7$, then the associative planes are those on which the linear functional φ assumes its maximum value 1, and constitute a "face" of the Grassmannian $\widetilde{Gr}_3(\mathbb{R}^7)$. This property means that the φ is a *calibration form* of $\mathbb{R}^7$, a notion introduced by Harvey and Lawson [HL], and shown to be extremely significant for the study of minimal varieties. See also [HM].

The roots of the Lie algebra $\mathfrak{g}_2$ can be read off from (11.2); they are

$$
\begin{array}{ccc}
\boxed{(2,0)} & (3,\sqrt{3}) & (3,-\sqrt{3}) \\
(-2,0) & (1,\sqrt{3}) & (1,-\sqrt{3}) \\
(0,2\sqrt{3}) & (-1,\sqrt{3}) & (-1,-\sqrt{3}) \\
(0,-2\sqrt{3}) & \boxed{(-3,\sqrt{3})} & (-3,-\sqrt{3}).
\end{array}
$$

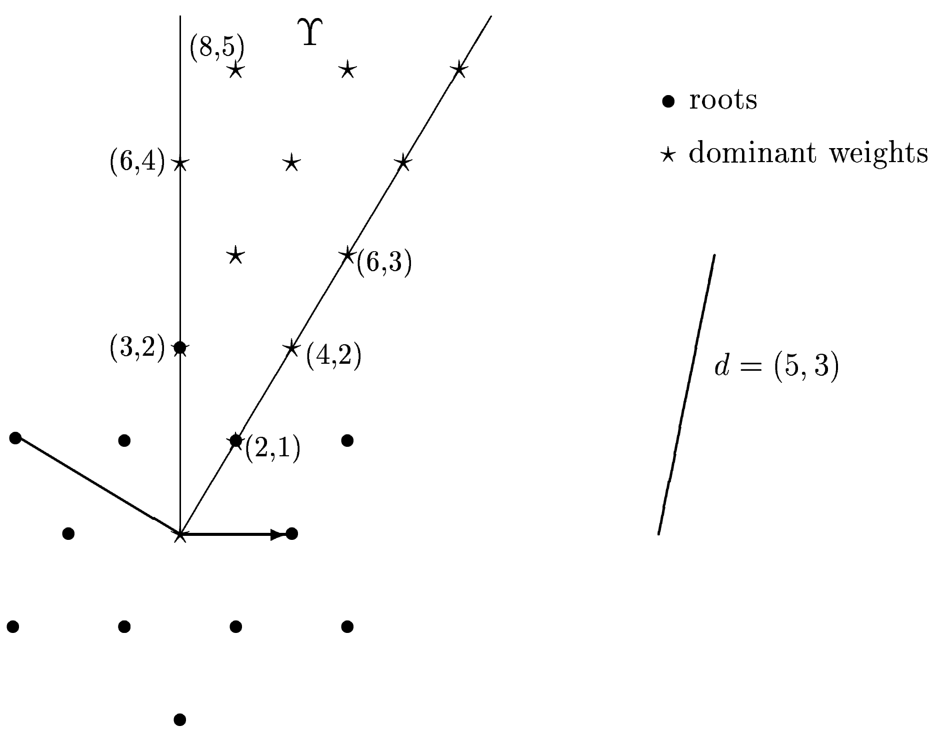
To obtain this list, we have taken each weight relative to $SO(4)$, interchanged the coordinates, and multiplied the new second one by $\sqrt{3}$; for instance, the highest weight $(1,3)$ of $\Delta^{1,3}$ became $(3,\sqrt{3})$. The $\sqrt{3}$ ensures that the resulting points have rotational symmetry about the origin in $\mathbb{R}^2$, which can be identified with the Lie algebra $\mathfrak{t}$ of a maximal torus of G_2.

To return to integral coordinates, it is customary to express the roots as linear combinations of two simple ones, which we have boxed. In the new coordinates, the roots become

$\boxed{(1,0)}$	$(3,1)$	$(0,\text{-}1)$
$(\text{-}1,0)$	$(2,1)$	$(\text{-}1,\text{-}1)$
$(3,2)$	$(1,1)$	$(\text{-}2,\text{-}1)$
$(\text{-}3,\text{-}2)$	$\boxed{(0,1)}$	$(\text{-}3,\text{-}1)$.

The following diagram shows the fundamental Weyl chamber (meant to subtend an angle of 30°), which is given by $\Upsilon = \{(a,b) : \frac{3}{2}b < a < 2b\}$.

11.3 Figure Roots and weights of G_2



11.4 Lemma [Bon$_1$] *There are isomorphisms*

$$\textstyle\bigwedge^2\mu \cong \mathfrak{g}_2 \oplus \mu$$
$$\textstyle\bigwedge^3\mu \cong \mathbb{R} \oplus \mu \oplus \bigodot_0^2\mu,$$

in which the components are all irreducible representations of G_2.

Proof. The decomposition of $\bigwedge^3\mu$ is a direct consequence of (11.6), and the subspace of $\bigwedge^2\mu$ isomorphic to μ itself is spanned by interior products of vectors with the invariant 3-form φ. Irreducibility may be checked by applying **6.5** to the tensor product $\mu\otimes\mu$. The weights of μ are (0,0) and those of the inner hexagon of **11.2**, including the highest weight (2,1). One obtains

(2,1)	(4,2) ✓	(4,2)
(1,1)	(3,2) ✓	(3,2)
(-1,0)	(1,1) ×	
(-2,-1)	(0,0) ✓	(0,0)
(-1,-1)	(1,0) ×	
(1,0)	(3,1) ×	
(0,0)	(2,1) ✓	(2,1),

whence

$$\begin{aligned} \mu\otimes\mu = (2,1)\otimes(2,1) &\cong (4,2)\oplus(3,2)\oplus(2,1)\oplus(0,0) \\ &\cong \textstyle\bigodot_0^2\mu\oplus\mathfrak{g}_2\oplus\mu\oplus\mathbb{R} \end{aligned} \tag{11.8}$$

does indeed have four irreducible components. Incidentally, the $SO(7)$-components of $\bigodot^k\mu$ remain irreducible under G_2, and $(2k,k)$ is the highest weight of the usual primitive subspace $\bigodot_0^k\mu$. □

$\mathbf{G_2}$ as a structure group

Suppose that φ now denotes a 3-form on a 7-dimensional manifold M, such that the stabilizer of φ is isomorphic to G_2 at each point. Recall that this is an *open* condition on φ. The G_2-structure allows us to identify each tangent or cotangent space of M with the representation μ. According to **5.3**, M has a distinguished G_2-invariant 4-form Ω. Up to a constant multiple, Ω can be identified with $*\varphi$, where $*$ denotes the star operator relative to the orientation and Riemannian metric determined by the inclusion $G_2 \subset SO(7)$ and so itself depends on φ. In the orthonormal notation of **11.1**,

$$\begin{aligned} *\varphi \;=\; & E^1E^2E^3E^4 - (E^1E^2 - E^3E^4)E^6E^7 \\ & -(E^1E^3 - E^4E^2)E^7E^5 - (E^1E^4 - E^2E^3)E^5E^6. \end{aligned}$$

11.5 Lemma [FG] *The holonomy group of the Riemannian metric induced by φ is contained in G_2 if and only if $d\varphi = 0 = d*\varphi$.*

Proof. From **2.2**, the obstruction $\nabla\varphi$ to the holonomy reduction has values in the space $\mu\otimes\mathfrak{g}_2^\perp$, which is isomorphic to (11.8). Since ∇ is torsion-free, $d\varphi$ and $d*\varphi$ are determined by $\nabla\varphi$ by means of respective $SO(7)$-equivariant linear maps

$$\begin{aligned} \partial\colon \mu\otimes\textstyle\bigwedge^3\mu &\longrightarrow \textstyle\bigwedge^4\mu \cong \bigwedge^3\mu \\ \partial^*\colon \mu\otimes\textstyle\bigwedge^3\mu &\longrightarrow \textstyle\bigwedge^5\mu \cong \bigwedge^2\mu. \end{aligned}$$

With the aid of **11.3**, it is a straightforward business to verify that these maps are both surjective, so that $d\varphi = 0 = d*\varphi$ implies $\nabla\varphi = 0$.

Incidentally, the fact that $\mu\otimes\mathfrak{g}_2^\perp$ has a unique μ-component implies that the μ-components of $d\varphi$ and $d*\varphi$, which can be identified with

$$\varphi\wedge *d\varphi, \quad \text{and} \quad *d*\varphi\wedge *\varphi$$

respectively, are proportional. An easy calculation using a trial φ with non-constant coefficients then shows that these two 6-forms are actually equal. □

The significance of G_2 as a structure group is enhanced by the fact that it is a maximal subgroup of $SL(7,\mathbb{R})$, a consequence of the irreducibility of $\bigodot_0^2\mu$. This contrasts with the complex and quaternionic situations; nevertheless in the present case there also exist natural conditions to impose that are less restrictive than the full holonomy reduction. Given a G_2-structure, two obvious conditions are that $d\varphi = 0$ ("symplectic") or $d*\varphi = 0$ ("cosymplectic", with apologies to those who use that expression for certain types of contact manifolds).

This approach was developed by Fernández and Gray [FG], and has recently resulted in an example of a *compact* parallelizable 7-manifold M with a symplectic

G_2-structure [F$_2$]. This manifold has a non-zero element $[\varphi] \in H^3(M, \mathbb{R})$, but none-the-less cannot carry a metric with holonomy in G_2, because if it did, the evaluated first Pontrjagin class

$$\left\langle p_1 \cup [\varphi], [M] \right\rangle = \frac{1}{8\pi^2} \int_M \operatorname{tr}(R \wedge R) \wedge \varphi$$

would have to be non-zero. This assertion is a consequence of the fact that the bilinear form b on $\mathfrak{g}_2 \subset \bigwedge^2 \mu$ defined by $\alpha \wedge \beta \wedge \varphi = b(\alpha, \beta)\varphi \wedge *\varphi$ is negative-definite. Further topological consequences of holonomy reductions have been studied by Bryant and Harvey [BH$_2$].

11.6 Figure Components of the G_2-structure function $\nabla\varphi$

$\mathfrak{g}_2$	$\mathbb{R}$
μ	$\bigodot^2_0 \mu$

0	0
0	0

G_2 Holonomy

0	?
0	?

Cosymplectic

?	0
0	0

Symplectic

The cosymplectic property is a type of integrability condition akin to self-duality for a 4-manifold, and is one of those conditions that is detected by the existence of a subcomplex of the de Rham complex. Bearing in mind **3.7** and **9.9**, we consider $A^0M = M \times \mathbb{R}$, $A^1M = T^*M$, and let A^2M, A^3M be the subbundles of $\bigwedge^2 T^*M$, $\bigwedge^3 T^*M$ with fibres $\mathfrak{g}_2^\perp$, $\mathbb{R}\varphi$ respectively. In addition, let D denote exterior differentiation followed by the appropriate projection.

11.7 Lemma *If M has a cosymplectic G_2-structure, then*

$$0 \to \Gamma(A^0M) \xrightarrow{D=d} \Gamma(A^1M) \xrightarrow{D} \Gamma(A^2M) \xrightarrow{D} \Gamma(A^3M) \to 0$$

is a complex.

Proof. If $(A^2M)^\perp$ denotes the subbundle of $\bigwedge^2 T^*M$ with fibre $\mathfrak{g}_2$, then the composition $D^2\colon A^1M \to A^3M$ vanishes if and only the composition $(A^2M)^\perp \to A^3M$ of d with orthogonal projection is zero. If α is a section of $(A^2M)^\perp$, then $\alpha \wedge *\varphi = 0$, and the second composition can be identified with the homomorphism

$$\alpha \;\mapsto\; d\alpha \wedge *\varphi = -\alpha \wedge d * \varphi,$$

which measures the component of $d * \varphi$ proportional to α. □

The operators of the resulting complex can be thought of as generalizations of *grad*, *div* and *curl*, whose existence is a consequence of the cross product (11.7). Other readers will be happier in the knowledge that the rearranged two-step complex

$$\Gamma(A^0M \oplus A^2M) \longrightarrow \Gamma(A^1M \oplus A^3M)$$

defines the Dirac operator of the 7-dimensional manifold M.

11.8 Proposition [Al$_1$] *The space of curvature tensors $\Re^{G_2}$ on a manifold M with holonomy contained in G_2 is isomorphic to the module with highest weight $(6,4)$, and M has zero Ricci tensor.*

Proof. Labelling modules by highest weights, a decomposition of $\mathfrak{g}_2 \otimes \mathfrak{g}_2$ and some dimension counting along the lines of **11.4** gives

$$\begin{aligned} \textstyle\bigodot^2 \mathfrak{g}_2 &\cong (6,4) \oplus \textstyle\bigodot^2_0 \mu \oplus \mathbb{R} \\ \textstyle\bigwedge^2 \mathfrak{g}_2 &\cong \mathfrak{g}_2 \oplus (6,3), \end{aligned}$$

and it is curious that both the modules irreducible (6,4),(6,3) have 77 dimensions. The remaining components of $\bigodot^2 \mathfrak{g}_2$ inject into $\bigwedge^4 \mu$. The fact that (6,4) is not a summand of $\bigodot^2 \mu$ implies the vanishing of the Ricci tensor. □

Further representation-theoretic calculations involving the second Bianchi identity reveal that the covariant derivative ∇R of the curvature tensor of M takes values in the module $(7,5)$. More generally, each higher covariant derivative $\nabla^k R$ is completely determined by its components in the module $(6+2k, 4+k)$ together with the derivatives of lower order, just like in **10.2**; once again this an effective illustration of the tight conditions imposed by a holonomy reduction. It is also relevant to the abstract existence techniques described by Bryant in [Br$_2$].

Let $\mathcal{V}M$ denote the subbundle of $\bigwedge^3 T^*M$ on a 7-manifold M, with fibre $\mathcal{V} \cong GL(7,\mathbb{R})/G_2$ consisting of positive non-degenerate 3-forms. To establish the existence of a metric with holonomy equal to G_2, one needs a section φ of $\mathcal{V}M$ for which (i) **11.5** holds identically, and (ii) at some point $m \in M$, the curvature tensor $R \in \mathfrak{R}^{G_2}$ satisfies $\mathrm{Im}(R) = \mathfrak{g}_2$. The higher curvature components enter as obstructions to extending the k-jet at m of a section of $\mathcal{V}M$ with $d\varphi$ and $d*\varphi$ zero to order $k-1$ to a $(k+1)$-jet with the same derivatives zero to order k. In [Br$_2$], Cartan-Kähler theory is used to prove that any such jet $j^2_m(\varphi)$ with $k=2$ can in fact be extended to a local solution φ of (i). The curvature tensor $R|_m$ at m will depend only on $j^2_m(\varphi)$ and can readily be preassigned so as to satisfy (ii).

The group G_2 acts transitively on the sphere $S^6 \subset \mu$, with stabilizer $SU(3)$. This well-known fact was, after all, the basis of the inclusion of G_2 as a possible holonomy group. From **11.1**, the invariant 3-form can be written

$$\varphi = \zeta + \omega \wedge E^7, \tag{11.9}$$

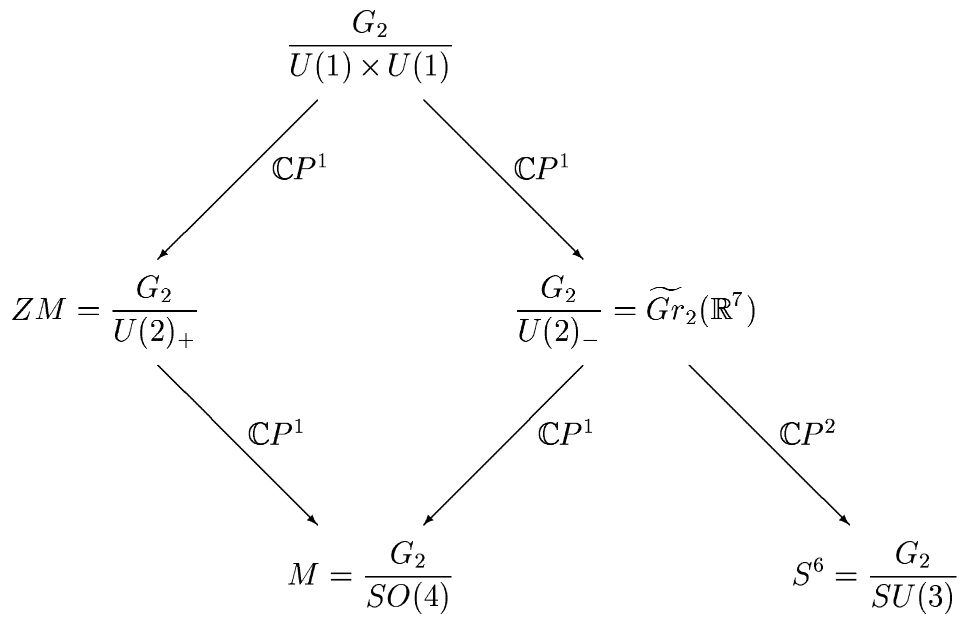
where

$$\begin{aligned} \omega &= E^1E^4 - E^2E^3 + E^5E^6 \\ \zeta &= \mathrm{Re}\{(E^1 + iE^4)(E^2 - iE^3)(E^5 + iE^6)\}. \end{aligned}$$

When G_2 acts on μ, the stabilizer of E^7 must act on its orthogonal complement as an 8-dimensional subgroup of $SU(3)$, that is $SU(3)$ itself.

Some insight into the relationship between the two subgroups $SO(4)$ and $SU(3)$ of G_2 is provided by the array below of maps between G_2 spaces. The upper three spaces are all orbits of the adjoint representation, and accordingly are equipped with natural complex structures, for which the appropriate fibres are complex submanifolds in the twistor space tradition. These spaces have been exploited by Musso [Mus] to study minimal surfaces in $M = G_2/SO(4)$, which behaves in some respects like a 4-dimensional Riemannian manifold.

11.9 Figure Fibrations involving G_2



Metrics with holonomy group $\mathbf{G_2}$

The formula (11.9) describes the special Hermitian structure of S^6, with E^7 playing the role of a constant 1-form on $\mathbb{R}^7$. The action of $SU(3)$ on the real 6-dimensional vector space $\mathbb{R}^6 = [\![\lambda^{1,0}]\!]$ gives rise to a real 2-dimensional space $[\![\lambda^{0,3}]\!]$ of invariant 3-forms, which contains ζ in the above situation. One can choose $\alpha \in [\![\lambda^{0,3}]\!]$ orthogonal to ζ in such a way that the following differential relations hold:

$$\begin{array}{ccccccc} \omega & \overset{d}{\longmapsto} & \zeta & \overset{d}{\longmapsto} & 0 & & \\ & & \alpha & \longmapsto & \omega^2 & \overset{d}{\longmapsto} & 0. \end{array} \tag{11.10}$$

These relations are forced on us by the $SU(3)$-invariance, and the fact that S^6 is nearly Kähler [FI], as defined at the end of chapter 3.

The twistor spaces

$$ZS^4 = \frac{SO(5)}{U(2)} = \mathbb{C}P^3,$$
$$Z\mathbb{C}P^2 = \frac{SU(3)}{T} = F^3,$$

endowed with the non-integrable almost complex structure J_2, are two other examples of homogeneous nearly-Kähler 6-manifolds. In fact they are *3-symmetric spaces*, being equipped with a 3-fold symmetry s_m about each point m, for which $J_2 = (2(s_m)_* + 1)/\sqrt{3}$, and are included in the classification of Wolf and Gray [WG]. The metrics in question are the normal ones induced from the bi-invariant metric on $SO(5)$ and $SU(3)$, and are in fact Einstein. This fact is justified independently by theorems in [G$_3$],[WZ] and [FGr], and motivated, by analogy with $\mathbb{R}^7 - \{0\}$, a search for metrics with holonomy G_2 on the total spaces

$$\mathbb{C}P^3 \times \mathbb{R}^+ = \Lambda^2_- S^4 - i_0(S^4), \qquad F^3 \times \mathbb{R}^+ = \Lambda^2_- \mathbb{C}P^2 - i_0(\mathbb{C}P^2). \tag{11.11}$$

minus their zero sections.

11.10 Theorem *If M^4 is a self-dual Einstein manifold, there exists a Riemannian metric with holonomy group contained in G_2 on a domain of the total space of $\Lambda^2_- M$.*

Proof. The list **7.2** is tailor-made for the task at hand. In analogy to (11.4), set

$$\begin{aligned} &E^1 = u\,\pi^* e^1,\ E^2 = u\,\pi^* e^2,\ E^3 = u\,\pi^* e^3,\ E^4 = u\,\pi^* e^4,\\ &E^5 = v b^1,\ E^6 = v b^2,\ E^7 = v b^3, \end{aligned} \tag{11.12}$$

where $\{e^i\}$ is the local orthonormal basis of 1-forms for which $\omega^1 = e^1e^2 - e^3e^4$ etc., and $u = u(\rho)$, $v = v(\rho)$ are functions of the radius squared ρ. Wherever u and v are non-zero, (11.12) determines a G_2-structure with

$$\begin{aligned} \varphi &= u^2 v d\tau + \tfrac{1}{6} v^3 \beta,\\ * \varphi &= u^4 \vartheta - \tfrac{1}{2} u^2 v^2 \gamma. \end{aligned}$$

Applying **7.3** gives

$$\begin{aligned} d\varphi &= (u^2v)' d\rho d\tau + \tfrac{1}{6} v^3 d\beta\\ &= \Big((u^2v)' - \tfrac{1}{24} v^3 t\Big) d\rho d\tau;\\ d * \varphi &= (u^4)' \vartheta d\rho - \tfrac{1}{2}(u^2v^2)' \gamma d\rho - \tfrac{1}{2} u^2 v^2 d\gamma\\ &= \Big((u^4)' - \tfrac{1}{6} u^2 v^2 t\Big) \vartheta d\rho - \tfrac{1}{2}(u^2v^2)' \gamma d\rho. \end{aligned}$$

To solve $d\varphi = 0 = d * \varphi$, we must first set uv equal to a constant k to kill off the coefficient of $\gamma d\rho = \frac{2}{3}\beta\tau$. The vanishing of the two remaining coefficients is then equivalent to the single equation $6(u^4)' = k^2t$; this compatibility, at first sight surprising, results from remarks made at the end of the proof of **11.5**. The final solution is

$$u = \left(\tfrac{1}{6}k^2t\rho + \ell\right)^{\frac{1}{4}}, \quad v = k\left(\tfrac{1}{6}k^2t\rho + \ell\right)^{-\frac{1}{4}}, \tag{11.13}$$

where ℓ is another constant. The form φ is non-degenerate provided that $\rho > -6\ell/k^2t$ (respectively $\rho < -6\ell/k^2t$) if $t > 0$ (respectively $t < 0$). The result now follows from **11.5**. □

The simplest-looking solution occurs when $t > 0$, when we may choose k so that $\frac{1}{6}k^2t = 1$ and take $\ell = 0$ to give a G_2-structure on $\Lambda^2_- M$ minus its zero section, which as in (11.11) equals $ZM \times \mathbb{R}^+$, where ZM is the hypersurface given by $\rho = 1$. Reverting to the radial parameter $r = \rho^{1/4}$, the associated Riemannian metric is

$$\begin{aligned} g_0 &= r^2 \sum_{i=1}^{4} e^i \otimes e^i + 6t^{-1}r^{-2} \sum_{i=1}^{3} b^i \otimes b^i \\ &= r^2\pi^* g^M + 6t^{-1}\left(4dr^2 + r^2 g^Z\right) \\ &= 24t^{-1}dr^2 + r^2 g^{ZM}, \end{aligned} \tag{11.14}$$

where Z denotes the 2-sphere, and $g^{ZM} = g^M + 6t^{-1}g^Z$ is a metric on ZM, analogous to the flat conical metric on $S^6 \times \mathbb{R}^+$.

The metric g^{ZM} involves a different scaling to the standard Kähler Einstein metric that exists on the twistor space when $t > 0$, and for $M = S^4$ or $\mathbb{C}P^2$ it coincides with the nearly Kähler Einstein one alluded to above. In contrast to **7.5**, the 2-form defined by g^{ZM} and the almost complex structure J_2, by means of (3.8), equals

$$\omega \;=\; 2(e^1e^2 - e^3e^4) + 6t^{-1}2b^2b^3 \;=\; 2\tau + 6t^{-1}\sigma$$

at the point $x \in ZM$ with $a^1 = 1$, $a^2 = 0 = a^3$ (see (7.18)). Using **7.2** and (7.16), observe that on ZM,

$$\begin{aligned} \omega^2 = \omega \wedge \omega &= 4\tau^2 + 24t^{-1}\sigma\tau \\ &= 24t^{-1}(\gamma - \tfrac{1}{3}t\vartheta) \\ &= 24t^{-1}d\alpha, \end{aligned}$$

which is consistent with (11.10), where α is the imaginary part of the canonical $(3,0)$-form (7.21).

The almost Hermitian manifold (ZM, J_2, g^{ZM}) is the one determined by the ambient G_2-structure in the sense of Calabi [Ca$_2$]. Moreover, its holomorphic curves, which parametrize minimal surfaces in M [ES], give rise to 3-dimensional submanifolds of $\Lambda^2_- M$ whose tangent spaces are associative with respect to the G_2-invariant cross product.

To obtain a *complete* metric with holonomy in G_2, we must suppose that M is itself complete and has $t > 0$; from **7.8** this means that M is either S^4 or $\mathbb{C}P^2$. Then taking $\frac{1}{6}k^2 t = 1$ and $\ell = 1$ in (11.13) gives the metric

$$g_1 = (\rho+1)^{\frac{1}{2}}\pi^* g^M + 6t^{-1}(\rho+1)^{-\frac{1}{2}}\sum_{i=1}^{3} b^i \otimes b^i,$$

which is asymptotic to (11.14) as $\rho = r^4 \to \infty$. Incredibly, this metric has exactly the same exponents as the 4-dimensional Eguchi-Hanson metric (8.7), and is complete for the same reason.

The question of whether all the metrics of **11.10** have holonomy group H *equal* to G_2 is more delicate. What we do know from **10.7** is that if H is a proper subgroup of G_2, then the metric is reducible, and (at least if M is simply-connected) one can show that there must be a non-zero covariant constant vector field [Br$_2$]. Special arguments are then available for the cases $M = S^4$ or $\mathbb{C}P^2$, in which the metric is invariant by the groups $SO(5)$ and $SU(3)$ respectively. Indeed, there can be no non-trivial representation of these groups on the space of covariant constant vector fields, which would otherwise be ridiculously large. On the other hand, no subspace of $T_x(\Lambda^2_- M)$ can be invariant by the respective isotropy groups $U(2)$, $U(1)\times U(1)$. One deduces

11.11 Corollary [BS] *The total spaces of $\Lambda^2_- S^4$ and $\Lambda^2_- \mathbb{C}P^2$ have Ricci-flat complete metrics with holonomy group equal to G_2.*

By construction, the respective groups $SO(5)$ and $SU(3)/\mathbb{Z}_3$ of isometries of the base lift to isometry groups of these Ricci-flat metrics. An examination of the remaining Weyl curvature tensor shows that there are no other isometries of the total space apart from multiplication by -1 on the fibres. In addition, we remark that under the weaker hypothesis that M is self-dual, the total space of $\Lambda^2_- M$ has a cosymplectic G_2-structure.

The above theory concerning the existence of the G_2-structures has its origins in the splitting of the 7-dimensional G_2-module relative to the subgroup $SO(4)$. If we reduce further to the kernel $SU(2)$ of the homomorphism $SO(4) \to \mathrm{Aut}(\Lambda^2_+)$, then (11.3) becomes

$$\mu = [\mathbb{C}^2 \otimes \sigma] \oplus [\sigma^2], \tag{11.15}$$

where $[\sigma^2]$ is the basic representation of $SU(2)/\mathbb{Z}_2 = SO(3)$, whose complexification is the symmetric square of the $SU(2)$-module σ. Such spinor terminology will be used extensively in the next chapter, but we indicate briefly here how it leads to another example of a G_2-metric, referring the reader to [BS] for more information.

The splitting (11.15) is realized geometrically on the total space of the *spin bundle* with fibre σ over an oriented Riemannian 3-manifold N. The summand $[\mathbb{C}^2 \otimes \sigma]$ is just a fancy way of expressing the real space underlying the fibre σ, and $[\sigma^2]$ is identified with a typical tangent space to N. It is now possible to formally repeat the construction **7.2** of invariant forms, although the roles of base and fibre become interchanged. In particular, a 3-form with stabilizer G_2 is formed by adding the volume form on N with a canonical 3-form φ formed from wedging 2-forms on the fibre with 1-forms on N. However, there is little flexibility in the curvature tensor of N and the equations $d\varphi = 0 = d * \varphi$ constrain N to have constant curvature. When $N = S^3$, one can again arrange for the resulting Riemannian metric to be complete with holonomy group equal to G_2; it is then defined on a manifold diffeomorphic to $S^3 \times \mathbb{R}^4$, and is linked to an Einstein metric on $S^3 \times S^3$.

Recall that the manner in which G_2-structures have been constructed by breaking up each tangent space into the direct sum of subspaces of dimension 3 and 4 had its origin in the existence of the symmetric Lie algebra (11.2). This has a further significance. Namely, with a few sign changes, the examples can be modified to produce pseudo-Riemannian metrics of signature $(3,4)$ and holonomy group G_2^*.

12 The Holonomy Group Spin(7)

It seems fair to include a definition of the groups $Spin(n)$ in this chapter, so we begin with a summary of the relevant theory of Clifford algebras, which will motivate some of the constructions which follow. However, the reader can freely skip this section, since our subsequent treatment of $Spin(7)$ will be largely independent of the general theory, being based instead on the fundamental 4-form built up from invariants of the group $SU(4)$, and subsequently G_2.

Although it helps to understand G_2 before tackling $Spin(7)$, there are certain features that make it relatively easy to recognize metrics on 8-dimensional manifolds whose holonomy is contained in $Spin(7)$. We discuss an example arising from an isotropy irreducible homogeneous space, which was in fact the first metric constructed with an exceptional holonomy group. Finally, we return to the concept of self-duality to construct $Spin(7)$-structures on the spin bundle of 4-dimensional manifold with self-dual and Einstein curvature.

Clifford algebras

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for the standard positive definite inner product g on $\mathbb{R}^n$. The Clifford algebra C_n is the algebra over $\mathbb{R}$ generated by an identity $\mathbf{1}$ and the symbols $e_1, \ldots, e_n$ subject to relations

$$\begin{aligned} e_r^2 &= -\mathbf{1}, \\ e_r e_s &= -e_s e_r, \quad r \neq s. \end{aligned}$$

The feature which can be used to characterize C_n in a universal sense is the inclusion $i\colon \mathbb{R}^n \hookrightarrow C_n$ with the property that $(i(x))^2 = -\|x\|^2\mathbf{1}$.

As a vector space, C_n has a basis consisting of the 2^n elements

$$\mathbf{1}, \quad e_{r_1} e_{r_2} \ldots e_{r_k}, \qquad r_1 < r_2 < \cdots < r_k, \quad 1 \leqslant k \leqslant n. \tag{12.1}$$

In fact the subspace generated by these elements for fixed k is independent of the original choice of basis, and provides a *natural* vector space isomorphism between C_n and the exterior algebra $\bigoplus\limits_0^n \bigwedge^k \mathbb{R}^n$. In these terms, C_n can be defined by

$$xy = x \wedge y + x \lrcorner y, \quad x \in \bigwedge^k \mathbb{R}^n, \ y \in \bigwedge^\ell \mathbb{R}^n, \ k < \ell \tag{12.2}$$

where $x \lrcorner y \in \bigwedge^{\ell-k}\mathbb{R}^n$ is a suitably defined interior product.

Of particular importance is the $\mathbb{Z}_2$-graded structure defined by

$$\begin{aligned} C_n^+ &= \{x_1x_2\dots x_{2k} : x_r \in \mathbb{R}^n\} \cong \bigoplus_{\text{even}} \Lambda^i, \\ C_n^- &= \{x_1x_2\dots x_{2k-1} : x_r \in \mathbb{R}^n\} \cong \bigoplus_{\text{odd}} \Lambda^i. \end{aligned}$$

Let $Spin(n)$ denote the subset of C_n^+ consisting of all even products $x_1x_2\dots x_{2r-1}x_{2r}$ of elements of $\mathbb{R}^n$ with each $\|x_i\|$ equal to 1. Then $Spin(n)$ is a closed subgroup of the group of all invertible elements of C_n, and contains both 1 and -1. If x is an element of $\mathbb{R}^n \subset C_n$ of norm one, then

$$\begin{aligned} -xyx^{-1} &= (yx + 2g(x,y))x^{-1} \\ &= y - 2g(x,y)x \\ &= r_x(y) \end{aligned}$$

is the reflection of y in the hyperplane perpendicular to x. The following is well-known:

12.1 Proposition *The mapping* $: x \mapsto r_x$ *induces an epimorphism* π *of* $Spin(n)$ *onto the special orthogonal group* $SO(n)$ *with kernel* $\{1,-1\}$, *and* $Spin(n)$ *is simply-connected for* $n \geqslant 3$.

For example, if $\alpha = \beta + e_r\gamma \in \ker\pi$ is written as a linear combination of the elements (12.1) so that β, γ do not involve e_r for some fixed r, then $\alpha e_r = e_r\alpha$ implies that $\gamma = 0$. Thus α is a multiple of the identity 1.

The group $Spin(3) \cong Sp(1)$ contains the unit elements

$$e_2e_3 = i, \quad e_3e_1 = j, \quad e_1e_2 = k$$

of the subalgebra C_3^+, which is itself isomorphic to the quaternions $\mathbb{H}$. It is easy to check that

$$C_1 \cong \mathbb{C}, \quad C_2 \cong \mathbb{H}, \quad C_3 \cong \mathbb{H} \oplus \mathbb{H}, \quad C_4 \cong \mathbb{H}(2), \tag{12.3}$$

where $F(n)$ denotes the algebra of $n \times n$ matrices with entries in F, and that

$$C_{n+4} \cong C_n \otimes C_4. \tag{12.4}$$

The last isomorphism can be seen by taking an orthonormal basis $\{f_1, f_2, f_3, f_4\}$ of $\mathbb{R}^4$ in addition to the basis $\{e_i\}$ of $\mathbb{R}^n$; then the elements

$$1\otimes f_1,\ 1\otimes f_2,\ 1\otimes f_3,\ 1\otimes f_4,\ e_1\otimes f_1f_2f_3f_4,\ \ldots,\ e_1\otimes f_1f_2f_3f_4$$

constitute a basis of $\mathbb{R}^{n+4}$, and satisfy the generating relations for C_{n+4}.

A representation $\rho\colon Spin(n) \to \operatorname{Aut} V$ factors through $SO(n)$ exactly when its kernel contains -1; if this is not the case, ρ is called *spin*. We shall illustrate the ensuing theory with $n = 7$ and 8 (an historical reference is [Li]). Fix an orthonormal basis $\{e_1, \ldots, e_8\}$ of $\mathbb{R}^8 \subset C_8$, and consider C_7 to be the subalgebra generated by $e_1, \ldots e_7$. From (12.3),(12.4), and the fact that $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{H} \cong \mathbb{R}(4)$, we have

$$\begin{aligned} C_7 &\cong C_3\otimes C_4 \cong \mathbb{R}(8) \oplus \mathbb{R}(8),\\ C_8 &\cong C_4\otimes C_4 \cong \mathbb{R}(16). \end{aligned} \tag{12.5}$$

There is a canonical involution $\varepsilon = e_1e_2\ldots e_8$ in C_8 that is independent of the choice of basis and commutes with $Spin(8)$. Consequently, there is an eigenspace decomposition

$$\mathbb{R}^{16} = \Delta_+ \oplus \Delta_-,$$

with representations $\rho_\pm\colon Spin(8) \to \operatorname{Aut}\Delta_\pm$. Since ε anti-commutes with any element in $\mathbb{R}^8 \subset C_8$, there are also maps

$$\mathbb{R}^8\otimes\Delta_\pm \longrightarrow \Delta_\mp,$$

known as *Clifford multiplication.* The subspaces $(1 \pm e_8)\Delta_+$ are invariant by C_7, which accounts for the description (12.5), and the two $Spin(8)$-modules Δ_+, Δ_- are isomorphic to a common $Spin(7)$-module.

The choice an isometry between $\mathbb{R}^8 \subset C_8$ and Δ_+ enables us to lift the homomorphism ρ_+ to an outer automorphism $\widetilde{\rho}$ of $Spin(8)$:

$$\begin{array}{ccc} & & Spin(8)\\ & \overset{\widetilde{\rho}}{\nearrow} & \downarrow \pi\\ Spin(8) & \xrightarrow{\ \rho_+\ } & SO(8) \subset \operatorname{Aut}\Delta_+ \end{array} \tag{12.6}$$

It is known that $\widetilde{\rho}^3 = \mathbf{1}$, and $\widetilde{\rho}$ is the so-called *triality automorphism*. For example, the inclusion (10.9) of $Spin(7)$ in $Spin(9)$ can be described as $i_8 \circ \widetilde{\rho} \circ i_7$, where $i_n : Spin(n) \hookrightarrow Spin(n+1)$ is the usual inclusion [B_2].

Spin representations may be described in arbitrary dimensions using the methods above, but because of the erratic description of the first eight real Clifford algebras, one has to complexify to make a clean statement. For each $m \geqslant 2$, there exist distinct complex 2^m-dimensional irreducible $Spin(2m)$-modules $\Delta_\pm$, which restrict to a common $Spin(2m-1)$-module Δ. In its turn, Δ restricts to the $Spin(2m-2)$-module $\Delta_+ \oplus \Delta_-$.

One can be more explicit by reducing $Spin(2m)$ to the subgroup $SU(m)$, lifted from its standard inclusion in $SO(2m)$. There are isomorphisms

$$\begin{aligned} \Delta_+ &\cong \bigoplus_{\text{even}} \lambda^{k,0}, \\ \Delta_- &\cong \bigoplus_{\text{odd}} \lambda^{k,0}, \end{aligned} \qquad (12.7)$$

where as usual $\lambda^{k,0}$ denotes an exterior power of the basic representation of $SU(m)$ on $\mathbb{C}^m$. Indeed, if $x \in \lambda^{1,0} \oplus \lambda^{0,1}$, there is an action

$$\lambda^{k,0} \ni \alpha \quad \longmapsto \quad x^{1,0} \wedge \alpha + x^{0,1} \lrcorner \alpha \in \lambda^{k+1,0} \oplus \lambda^{0,k-1}$$

which extends to a representation of the Clifford algebra $C_{2m} \otimes_{\mathbb{R}} \mathbb{C}$ on (12.7), with multiplication as in (12.2).

Spin(7) as a structure group

Our description of G_2 in the preceding chapter was based on the subgroup $SO(4)$, which is the stabilizer of an appropriate symmetric space. An analogous description of $Spin(7)$ proceeds from the group $Spin(6)$, which is the stabilizer for the action of $Spin(7)$ on the 6-sphere S^6.

Another explanation of the fact **6.4** that $Spin(6)$ is isomorphic to the group $SU(4)$ of special unitary transformations of a complex 4-dimensional space $\lambda^{1,0}$ would not go amiss. In analogy with the real 4-dimensional case (7.3), $SU(4)$ commutes with an isomorphism $*\colon \lambda^{k,0} \to \lambda^{0,4-k}$. In particular $\lambda^{2,0} \cong \lambda^{0,2}$ is the complexification of a real 6-dimensional vector space $[\lambda^{2,0}]$, which gives rise to the required 2:1 homomorphism $SU(4) \to SO(6)$.

The 8-dimensional space $\lambda^{1,0} \oplus \lambda^{0,1}$ can in fact be identified with the restriction of the spin representation Δ of $Spin(7)$. Before verifying this, let

$$\{E^1 + iE^5, E^2 + iE^8, E^3 + iE^7, E^4 - iE^6\} \tag{12.8}$$

be a special unitary basis of $\lambda^{1,0}$ (labelled eccentrically as a result of conventions that had their origin in chapter 7), and consider the $SU(4)$-invariant 4-form

$$\begin{aligned}
\Omega &= \mathrm{Re}\Big[(E^1+iE^5)(E^2+iE^8)(E^3+iE^7)(E^4-iE^6)\Big] \\
&\qquad\qquad -\tfrac{1}{2}\Big[E^1E^5+E^2E^8+E^3E^7-E^4E^6\Big]^2 \\
&= (E^1E^2E^5-E^3E^4E^5+E^1E^3E^6-E^4E^2E^6+E^1E^4E^7-E^2E^3E^7+E^5E^6E^7)E^8 \\
&\qquad +E^1E^2E^3E^4-E^1E^2E^6E^7+E^3E^4E^6E^7-E^1E^3E^7E^5 \\
&\qquad\qquad +E^4E^2E^7E^5-E^1E^4E^5E^6+E^2E^3E^5E^6 \\
&= \varphi\wedge E^8 + *\varphi,
\end{aligned} \tag{12.9}$$

in the notation of the preceding chapter.

12.2 Lemma *The subgroup of $GL(8,\mathbb{R})$ leaving Ω invariant is isomorphic to $Spin(7)$, which acts transitively on S^7 with stabilizer G_2.*

Proof. Compared to **9.1** and **11.1**, we now have the luxury of two subgroups to play with simultaneously. Let H denote the stabilizer of ψ in $GL(8,\mathbb{R})$, and Δ the vector space $\mathbb{R}^8$ upon which it acts. By construction, H contains both $SU(4)$ and $G_2 \times \{e\}$, and the relationship between the three groups H, $SU(4)$ and G_2 is expressed in representation-theoretic terms by the formula

$$\Delta \qquad \cong [\![\lambda^{1,0}]\!] \qquad \cong \mu \oplus \mathbb{R}.$$

The Lie algebra $\mathfrak{h} \subset \mathfrak{sl}(8,\mathbb{R})$ can be dug out from amongst the spaces

$$\begin{aligned}
\textstyle\bigodot_0^2\Delta \qquad &\cong [\![\sigma^{2,0}]\!] \oplus \lambda_0^{1,1} &&\cong \textstyle\bigodot_0^2\mu \oplus \mu \oplus \mathbb{R}, \\
\textstyle\bigwedge^2\Delta \cong \mathfrak{so}(8) \qquad &\cong [\lambda^{2,0}] \oplus [\lambda^{2,0}] \oplus \mathfrak{su}(4) \oplus \mathbb{R} &&\cong \mathfrak{g}_2 \oplus \mu \oplus \mu.
\end{aligned}$$

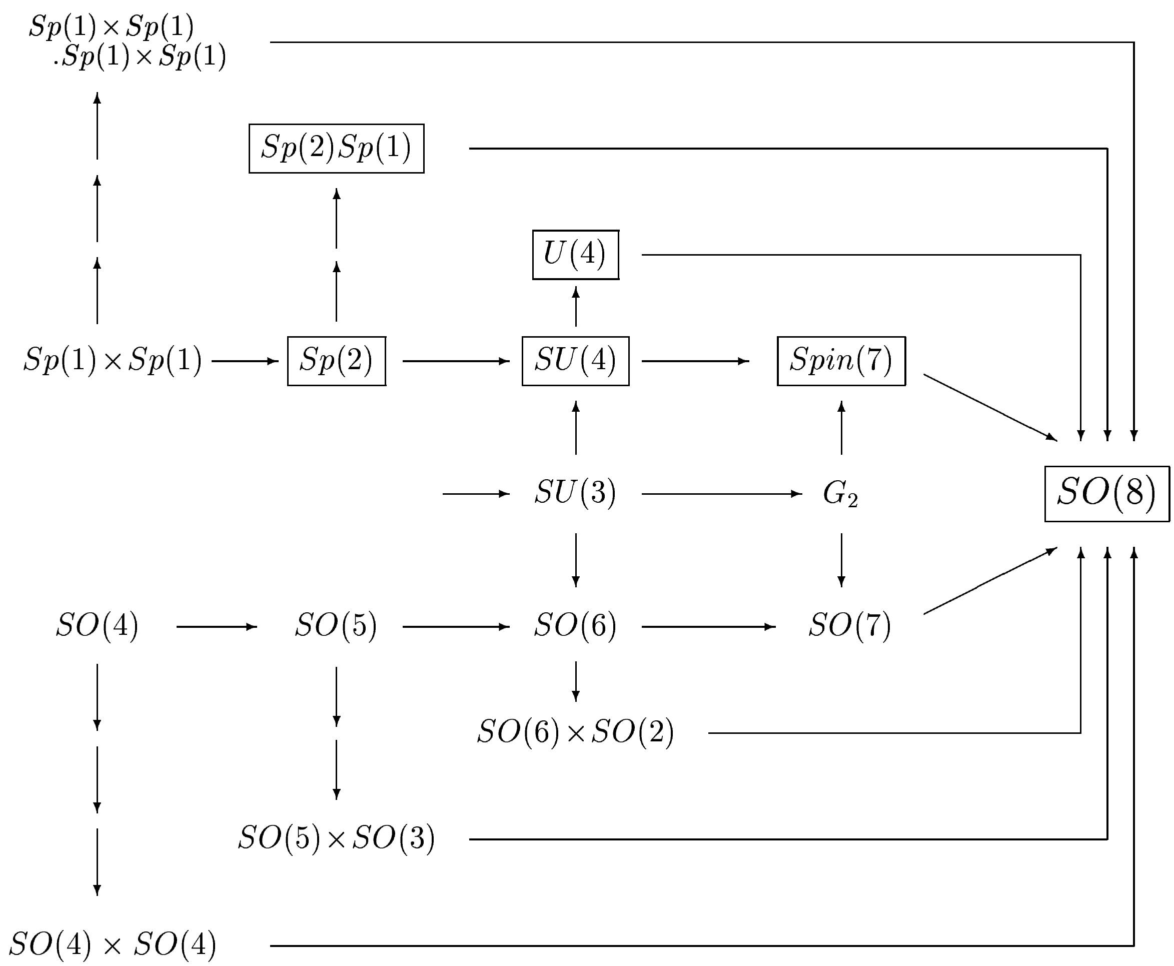
Indeed, a match of their dimensions yields

$$\mathfrak{h} \qquad \cong \mathfrak{su}(4) \oplus [\lambda^{2,0}] \qquad \cong \mathfrak{g}_2 \oplus \mu, \tag{12.10}$$

and shows that $\bigwedge^2\Delta$ has a 7-dimensional H-invariant summand which exhibits the required double covering $H \to SO(7)$. The two isomorphisms in (12.10) correspond to the homogeneous spaces $S^6 \cong Spin(7)/Spin(6)$ and $S^7 \cong Spin(7)/G_2$, the second of which is the orbit containing E^8 in Δ. $\square$

The richness of 8-dimensions is illustrated by the following display of subgroups of $SO(8)$, extending (6.10). The boxed ones are possible holonomy groups of irreducible Riemannian metrics; they are paired with more obvious subgroups acting reducibly, by means of the triality automorphism (12.6) which interchanges tangent and spin representations. Whereas the group $SO(8-k)$ is the subgroup of $SO(8)$ fixing k linearly independent tangent vectors, so the subgroups in the same row as $Spin(7)$ are characterized by the property that they fix k linearly independent spinors, where k ranges from 1 to 4. A unified treatment of the characterization of holonomy reductions by parallel spinors and forms is given in [W]; we exploited topological consequences of the above facts in chapter 8.

12.3 Figure Selected subgroups of $SO(8)$



Although we chose to build up the $Spin(7)$ form Ω from invariants of $SU(4)$ and G_2, another possibility is to start from the subgroup $Sp(2)$ leaving fixed "Kähler triple" $\omega^1, \omega^2, \omega^3$ of non-degenerate 2-forms. Whereas the 4-form $\sum_{i=1}^{3}(\omega^i)^2$ leads to the subgroup $Sp(2)Sp(1)$, its modification $(\omega^1)^2 + (\omega^2)^2 - (\omega^3)^2$ equals twice (12.9) in suitable coordinates, and therefore has stabilizer $Spin(7)$. This observation was used by Bryant and Harvey [BH_1] to pursue higher-dimensional generalizations of $Spin(7)$-geometry in a hyperkähler context.

A reduction of the principal frame bundle of an 8-manifold M to $Spin(7)$ is characterized by a 4-form Ω which is linearly equivalent at each point to (12.9). The inclusion $Spin(7) \subset SO(8)$ induces a Riemannian metric on M, and as usual, the failure of the holonomy to reduce to $Spin(7)$ is measured by $\nabla\Omega$.

We denote the 7-dimensional representation of $Spin(7)$ that factors through $SO(7)$ by Λ^1. The proof of **12.2** exhibited Λ^1 is a submodule of $\bigwedge^2\Delta$, which implies the existence of a homomorphism

$$m\colon \Lambda^1 \otimes \Delta \longrightarrow \Delta,$$

otherwise known as Clifford multiplication. This can be understood by extending the methods of **6.9** to cover $Spin(7)$ by allowing half-integral weights. The weights of Δ are then the eight triples $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$, and there are decompositions

$$\begin{aligned}
\Delta\otimes\Delta &= (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})\otimes(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})\\
&\cong (0,0,0) \oplus (1,0,0) \oplus (1,1,0) \oplus (1,1,1)\\
&\cong \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \Lambda^3;
\end{aligned}$$

$$\begin{aligned}
\Lambda^1\otimes\Delta &= (1,0,0)\otimes(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})\\
&\cong (\tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2}) \oplus (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})\\
&\cong \Delta' \oplus \Delta.
\end{aligned}$$

In particular, the $Spin(7)$-module $\Delta' = \ker m$ is irreducible, and Schur's lemma can be used to check that the composition

$$\Lambda^1\otimes\Delta \hookrightarrow \bigwedge^2\Delta\otimes\Delta \longrightarrow \bigwedge^3\Delta \qquad (12.11)$$

is an isomorphism. This striking fact allows us to identify $\nabla\Omega$ with $d\Omega$ so that

12.4 Lemma [Br$_2$] *The holonomy group of the Riemannian metric defined by the 4-form Ω is contained in $Spin(7)$ if and only if $d\Omega = 0$.*

The fact that $\nabla\Omega$ has only two components (commented on in [F$_1$]) contrasts with the situations we have encountered for other holonomy groups. Of course, the orbit $GL(8,\mathbb{R})/Spin(7)$ containing Ω is not open in $\bigwedge^4\Delta$; indeed the following result, easily deduced from remarks above, identifies its normal space at Ω with the space Σ_0^2 of traceless symmetric forms on $\mathbb{R}^7$.

12.5 Proposition *There are isomorphisms*

$$\begin{aligned}
&\textstyle\bigwedge^2\Delta \cong \Lambda^1 \oplus \Lambda^2,\\
&\textstyle\bigwedge^3\Delta \cong \Delta \oplus \Delta',\\
&\textstyle\bigwedge^4_+\Delta \cong \mathbb{R} \oplus \Lambda^1 \oplus \Sigma^2_0, \quad \bigwedge^4_-\Delta \cong \Lambda^3.
\end{aligned}$$

12.6 Corollary [Al$_1$] *The space $\mathfrak{R}^{Spin(7)}$ of reduced curvature tensors is isomorphic to the submodule W of* **6.9***, and any metric with holonomy group contained in $Spin(7)$ has zero Ricci tensor.*

Proof. The space $\mathfrak{R}^{Spin(7)}$ equals the kernel of

$$\textstyle\bigodot^2\mathfrak{so}(7) \hookrightarrow \bigodot^2(\bigwedge^2\Delta) \longrightarrow \bigwedge^4\Delta.$$

Combining **6.9** and **12.5** with Schur's lemma, it suffices to check that there exist 2-forms $\alpha, \beta, \gamma \in \mathfrak{so}(7) \subset \bigwedge^2\Delta$ for which $\alpha\wedge\alpha$, $\beta\wedge\beta$ are linearly independent elements of $\bigwedge^4_+\Delta$, and $\gamma\wedge\gamma$ is a non-zero element of $\bigwedge^4_-\Delta$. The vanishing of the Ricci tensor will follow from the fact that $\bigodot^2\Delta$ does not contain W.

The Lie algebra $\mathfrak{so}(7)$ of $Spin(7)$ contains $\mathfrak{su}(4)$, which can be identified with the space $[\lambda_0^{1,1}]$ of primitive $(1,1)$ forms relative to a Hermitian structure on Δ. Using the decomposition (8.12), α, β, γ are readily found inside $\mathfrak{su}(4)$. □

We shall not prove the corresponding statement for the subgroup $Spin(9)$ of $SO(16)$, namely that $\mathfrak{R}^{Spin(16)} \cong \mathbb{R}$, which forces any metric with holonomy group contained in $Spin(9)$ to be symmetric [Al$_1$],[BG]. However, an analysis of this case may be made by writing down an explicit matrix representation of the Clifford algebra C_9, or by decomposing the spin representation of $Spin(9)$ as $\mathbb{R}^{16} = \Delta\oplus\Delta$ with respect to the subgroup $Spin(7)$. Although the latter has three independent invariant elements in $\bigwedge^4(\Delta \oplus \Delta)$ it is easy to see that no combination of these is annihilated by $\mathfrak{so}(9) \cong \mathbb{R} \oplus \Lambda^1 \oplus \Lambda^1 \oplus \mathfrak{so}(7)$. This implies the existence of a symmetric Lie algebra

$$\mathfrak{f}_4 = \mathfrak{so}(9) \oplus \mathbb{R}^{16}. \tag{12.12}$$

Further verification is required to show that the non-trivial components of $\bigodot^2 \mathfrak{so}(9)$ all inject into $\bigwedge^4(\Delta \oplus \Delta)$.

A sporadic example

From **12.2**, the flat $Spin(7)$-structure on $\mathbb{R}^8$ induces on any hypersurface N a G_2-structure. If Ω is the associated 4-form on $\mathbb{R}^8$, then from (12.9), $i^*\Omega = *\varphi$, and so the resulting G_2-structure on N is cosymplectic (cf. **11.7**). The remaining two components of $d\varphi$ relate to the second fundamental form of N [Br$_1$]. For example the structure on the totally umbilic sphere S^7 must be invariant by the isotropy group G_2 itself, so

$$d\varphi = k * \varphi, \quad d * \varphi = 0, \tag{12.13}$$

where k is a non-zero constant on S^7. Actually, a suitable scaling of the metric allows one to set $k = 1$.

By analogy to **11.7**, one might predict that a G_2-structure satisfying (12.13), but whose metric is not isometric to S^7, will lead to a metric whose holonomy is contained in $Spin(7)$. Structures satisfying (12.13) are said to have "weak holonomy group" G_2, a notion introduced by Gray [G$_2$]; they have the property that parallel translation preserves the associative subspaces defined by the cross product (11.7). We shall exhibit such a structure on a non-standard homogeneous space $SO(5)/SO(3)$, but we must first discuss the representation theory of $SU(2)$.

The complex irreducible $SU(2)$-modules are precisely the k-fold symmetric tensor products $\sigma^k = \bigodot^k \sigma$ of the basic one $\sigma \cong \mathbb{C}^2$. The space σ^k may be regarded as that consisting of homogeneous polynomials of degree k in two variables, and multiplication of polynomials determines an $SU(2)$-invariant mapping $\sigma^k \otimes \sigma^\ell \to \sigma^{k+\ell}$. On the other hand, an invariant skew form $\bigwedge^2 \sigma \to \mathbb{C}$ extends to make contractions $\sigma^k \to \sigma^{k-2i}$ whenever $1 \leqslant i \leqslant [k/2]$. Using Schur's lemma it is now easy to deduce the "Clebsch-Gordan" formula

$$\sigma^k \otimes \sigma^\ell \cong \sum_{i=0}^{\min\{k,\ell\}} \sigma^{k+\ell-2i}, \tag{12.14}$$

which may be regarded as the simplest version of **6.2**.

Observe that σ^{2k} is the complexification of a real vector space $[\sigma^{2k}]$ equal to the fixed points of ε^{2k}, where ε corresponds to multiplication by the quaternion j on σ, which is invariant by $SU(2) = Sp(1)$. For example $[\sigma^2]$ determines the double covering $SU(2){=}Spin(3) \to SO(3)$, and as well as being the basic representation of $SO(3)$, it is also isomorphic to the adjoint representation via the usual vector cross product:

$$\mathfrak{su}(2) = \mathfrak{so}(3) \;\cong\; \bigwedge^2[\sigma^2] \;\stackrel{*}{\longrightarrow}\; [\sigma^2].$$

More generally, $[\sigma^{2k}] = \bigodot_0^k[\sigma^2]$ is the primitive component of

$$\bigodot^k[\sigma^2] \;\cong\; [\sigma^{2k}] \oplus [\sigma^{2k-2}] \oplus \cdots \oplus [\sigma^2] \oplus \mathbb{R}.$$

We now ask the question "is $[\sigma^{2k}]$ the isotropy representation of a homogeneous space?". This is the case when the bracket on $\mathfrak{so}(3)$ extends to make

$$\mathfrak{g} \;=\; \mathfrak{so}(3) \oplus [\sigma^{2k}] \;\cong\; [\sigma^2] \oplus [\sigma^{2k}]$$

into a Lie algebra by means of contractions

$$\bigwedge^2[\sigma^{2k}] \to [\sigma^2], \quad \bigwedge^2[\sigma^{2k}] \to [\sigma^{2k}]$$

arising from (12.14). The second of these projections is necessarily zero when k is even, so in this case $\mathfrak{g}$ would be automatically be a symmetric Lie algebra. The only point where the Jacobi identity remains in doubt involves the brackets $[x,[y,z]]$, where x, y, z all belong to $[\sigma^{2k}]$.

We quote the following result as a corollary of the classification, due independently to Manturov [Ma] and Wolf [W_4], of homogeneous spaces with irreducible isotropy action, tempting the reader to give a direct proof.

12.7 Proposition *$[\sigma^{2k}]$ the isotropy representation of a homogeneous space M if and only if k equals $1, 2, 3$ or 5, and $\mathfrak{g}$ is one of $\mathfrak{so}(4)$, $\mathfrak{su}(3)$, $\mathfrak{so}(5)$, $\mathfrak{g}_2$.*

The corresponding simply-connected spaces have the form

$$S^3 \cong SU(2) \cong \frac{SO(4)}{SO(3)}, \quad \frac{SU(3)}{SO(3)}, \quad \frac{SO(5)}{SO(3)}, \quad \frac{G_2}{SO(3)}.$$

The action of $SO(3)$ on the G_2-module μ determined by the last space must satisfy

$$\bigwedge^2\mu \cong \mathfrak{g}_2 \oplus \mu \cong [\sigma^2] \oplus [\sigma^{10}] \oplus \mu,$$

and the only solution is $\mu = [\sigma^6]$. In other words, the homomorphism

$$SO(3) \longrightarrow SO(7) \subset \mathrm{Aut}\,[\sigma^6], \tag{12.15}$$

which is itself the isotropy representation of $SO(5)/SO(3)$, factors through G_2.

12.8 Theorem [Br_3] *There exists a Riemannian metric with holonomy equal to* $Spin(7)$ *on the product* $\mathbb{R}^+ \times \dfrac{SO(5)}{SO(3)}$.

Proof. From (12.15) the principal $SO(3)$-bundle over $M = SO(5)/SO(3)$ with total space $SO(5)$ lies in a G_2-subbundle of the frame bundle. The resulting structure then equips M with with a 3-form φ and a Riemannian metric g^M. It is remarkable that the refinement

$$\begin{aligned} \mu\otimes\mu &\cong [\sigma^6]\otimes[\sigma^6] \\ &\cong [\sigma^{12}] \oplus [\sigma^{10}] \oplus [\sigma^8] \oplus [\sigma^6] \oplus [\sigma^4] \oplus [\sigma^2] \oplus \mathbb{R}, \end{aligned}$$

of (11.8) contains a *unique* $SO(3)$-invariant. The same is true of the space

$$\begin{aligned} \bigwedge^3[\sigma^6] &\cong \bigodot^3[\sigma^4] \\ &\cong [\sigma^{12}] \oplus [\sigma^8] \oplus [\sigma^6] \oplus [\sigma^4] \oplus \mathbb{R} \end{aligned}$$

of 3-forms on M, whose decomposition was deduced immediately from **11.4**.

In addition, $\bigwedge^2[\sigma^6] \cong \bigwedge^5[\sigma^6]$ contains no $SO(3)$-invariants, and $H^3(M,\mathbb{R}) = 0$ (see below), so we may deduce that φ satisfies (12.13). If t is a real coordinate, the 4-form

$$\Omega = \varphi\, dt - kt * \varphi$$

defines a $Spin(7)$-structure on $\mathbb{R}^+ \times M$ by (12.9), provided $kt < 0$. Since Ω is closed, the holonomy group of the corresponding metric $dt^2 + k^2t^2g^M$ is contained in $Spin(7)$, and the holonomy algebra $\mathfrak{h}$ must lie between $\mathfrak{so}(3)$ and $\mathfrak{so}(7) = \mathfrak{so}(3) \oplus [\sigma^6]\oplus[\sigma^{10}]$. However, $\mathfrak{g}_2 = \mathfrak{so}(3)\oplus[\sigma^{10}]$ is the only proper subalgebra whose holonomy representation restricts to the $SO(3)$-module $\mathbb{R} \oplus [\sigma^6]$. This is ruled out by showing that no non-zero multiple of dt can be parallel. Thus $\mathfrak{h} = \mathfrak{so}(7)$. □

The manifold $M = SO(5)/SO(3) = Sp(2)/SU(2)$ has an established history as an odd man out. Berger proved that, apart from the rank one symmetric spaces, there are only two normal homogeneous manifolds with everywhere strictly positive sectional curvature, one of which is M [Be$_3$]. He also showed that M is not homeomorphic to S^7, for although they share the same real cohomology, M has torsion in dimension 3. The positive curvature of M is a consequence of the algebraic fact that the Lie bracket of

$$\mathfrak{so}(5) = \mathfrak{so}(3) \oplus \mathfrak{m}$$

has the property that $[x, y] \neq 0$, whenever x, y are linearly independent elements of the subspace $\mathfrak{m} = [\sigma^6]$. This in turn follows from the fact that its $\mathfrak{m}$-component can be identified with the cross product $x \times y$ of (11.7).

The inclusion $SO(3) \subset SO(5)$ defining M induces an embedding

$$S^2 = \frac{SO(3)}{SO(2)} \longrightarrow \frac{SO(5)}{SO(4)} = S^4,$$

whose image is a *Veronese* surface in S^4. It follows that M parametrizes Veronese surfaces in S^4, which lift to holomorphic rational normal curves in the twistor space $\mathbb{C}P^3$ [ES]. Similarly, its isotropy irreducible partner $G_2/SO(3)$ parametrizes Veronese surfaces in S^6 which are holomorphic relative to the standard almost complex structure of S^6.

The metric of **12.8** has the same form as (11.14). Applying this cone construction to an arbitrary 7-manifold M with a G_2-structure satisfying the weak holonomy condition (12.13), one can use the fact that any $Spin(7)$-metric is Ricci-flat to deduce that M is Einstein, a fact proved independently in [G$_2$]. As far as our original choice of M is concerned, any isotropy irreducible space is automatically Einstein, as all bilinear forms on the tangent space are proportional. In the final section, we shall produce an analogous metric, in which M is replaced by an isotropy *reducible* space $Sp(2)/Sp(1)$, namely the sphere S^7.

Spin(7)-metrics and self-duality

Let M be a self-dual Einstein 4-manifold M, as in the preceding chapter. This time we must also suppose that M is a *spin manifold*, which we recall means that its principal bundle P of oriented orthonormal frames lifts to a $Spin(4)$ or $SU(2) \times$

$SU(2)$-bundle $\widetilde{P}$. Of course, this is always true locally, but it general it requires the vanishing of the second Stiefel-Whitney class $w_2(M)$. We shall be working on the real 8-dimensional total space of the spin bundle

$$\sigma_- M \;=\; \widetilde{P} \times_{Spin(4)} \sigma_-,$$

where σ_- denotes the basic complex 2-dimensional vector space on which the second $SU(2)$ factor acts (see (11.1)).

For the purposes of calculation, we may choose a section $\omega \in \Gamma(U, \Lambda^2_+ M)$ of unit norm, defining a positively oriented orthogonal almost complex structure on the open set U. Suppose also that $\{e^1, e^2\}$ is a Hermitian basis of $(1,0)$-forms on U, with $\omega = -i(e^1\overline{e^1} + e^2\overline{e^2})$, which is consistent with the definition (3.6). Then there exist *real* forms c^i, E^i, on the total space $\sigma_- M$ such that

$$\begin{aligned} -i\pi^*(e^1\overline{e^1} - e^2\overline{e^2}) &= c^1 = E^1E^2 - E^3E^4, \\ \pi^*(e^1\overline{e^2} - e^2\overline{e^1}) &= c^2 = E^1E^3 - E^4E^2, \\ -i\pi^*(e^1\overline{e^2} + e^2\overline{e^1}) &= c^3 = E^1E^4 - E^2E^3; \end{aligned} \tag{12.16}$$

what we have done here is to write down explicitly the isomorphism $\lambda^{1,1}_0 M \cong \Lambda^2_- M$ ((7.7) with orientations reversed).

The complex symplectic form e^1e^2, of type $(2,0)$ relative to ω, defines an $SU(2)$-structure on U. With respect to this, e^1, e^2 may be regarded as sections of the spinor bundle according to the isomorphism $\sigma_- \cong \lambda^{1,0}$, which is really a special case of (12.7). The elements a^1, a^2 of the dual basis to $\{e^1, e^2\}$ may be regarded as complex-valued coordinates on $\sigma_- M$. Adapting the techniques of chapter 7, we define 1-forms

$$b^i \;=\; da^i + \sum_{j=1}^{2} a^j \pi^* \psi^j_i, \quad 1 \leqslant i \leqslant 2,$$

where $\nabla e^i = \sum_j \psi^i_j \otimes e^j$. The equations

$$\begin{aligned} i(b^1\overline{b^1} - b^2\overline{b^2}) &= f^1 = E^5E^8 - E^7(-E^6), \\ (b^1\overline{b^2} - b^2\overline{b^1}) &= f^2 = E^5E^7 - (-E^6)E^8, \\ i(b^1\overline{b^2} + b^2\overline{b^1}) &= f^3 = E^5(-E^6) - E^8E^7 \end{aligned} \tag{12.17}$$

define 1-forms $E^5, E^8, E^7, -E^6$ (labelled in accordance with (12.8)) and 2-forms f^1, f^2, f^3 on σ_-M. Observe that the 1-forms e^1, e^2 of (12.16) have been replaced by $\overline{b^1}, \overline{b^2}$ respectively, to account for the Hermitian duality.

Let $u = u(\rho)$, $v = v(\rho)$ be positive functions of the Hermitian norm squared $\rho = \sum_{i=1}^{2} |a^i|^2$. Comparing (12.16),(12.17) with (12.9), we see that

$$\Omega \;=\; -\tfrac{1}{6}u^2 \sum_{i=1}^{3} c^i c^i + uv \sum_{i=1}^{3} c^i f^i - \tfrac{1}{6}v^2 \sum_{i=1}^{3} f^i f^i \tag{12.18}$$

is a 4-form whose stabilizer is isomorphic to $Spin(7)$, and associated to the Riemannian metric

$$u \sum_{i=1}^{4} E^i \otimes E^i + v \sum_{i=5}^{8} E^i \otimes E^i. \tag{12.19}$$

We leave the reader to verify that the definition of Ω does not depend on our choice of basis, so that it is a globally-defined 4-form on the total space of the spin bundle.

12.9 Proposition *If M is a 4-dimensional self-dual Einstein spin manifold with non-zero scalar curvature t, then for some range of values of ρ, the functions u, v can be chosen so that* (12.19) *is a metric with holonomy group contained in* $Spin(7)$.

Proof. Note that $-\frac{1}{6}\sum c^i c^i$ and $-\frac{1}{6}\sum f^i f^i$ are just fancy ways of writing the pullback $\pi^*\vartheta$ of the volume form on M, and the corresponding "vertical" 4-form $-b^1\overline{b^1}b^2\overline{b^2}$. Now

$$db^i \;=\; \sum_j (b^j \pi^* \psi_i^j + a^j \pi^* \Psi_i^j),$$

where Ψ_i^j are the curvature 2-forms which satisfy

$$\Psi_i^j \;=\; \tfrac{1}{12} t (\overline{e^i} e^j)_0,$$

where the subscript $_0$ denotes the primitive component relative to ω. This is a complex analogue of (7.15), whence it may be proved directly.

For the purposes of computing derivatives, we are free to work at a fixed point $x \in \pi^{-1}(m)$ for which $a^2(x) = 0$ and $\nabla e^i|_m = 0$. In this case,

$$db^1|_x \;=\; a^1\psi_1^1 \;=\; -\tfrac{1}{24} i t a^1 c^1,$$

$$db^2|_x \;=\; a^1\psi_2^1 \;=\; -\tfrac{1}{24} t a^1 (c^2 + i c^3),$$

and

$$\begin{aligned} d\Omega|_x &= \mathrm{Re}\Big[(u^2)'d\rho.\pi^*\vartheta - \tfrac{1}{2}tuva^1\overline{b^1}\pi^*\vartheta + (uv)'d\rho\sum c^i f^i \\ &\qquad + \tfrac{1}{12}ta^1v^2\Big(ic^1\overline{b^1}b^2\overline{b^2} + (c^2+ic^3)b^1\overline{b^1b^2}\Big)\Big] \\ &= \Big((u^2)' - \tfrac{1}{4}tuv\Big)d\rho.\pi^*\vartheta + \Big((uv)' - \tfrac{1}{24}tv^2\Big)d\rho\sum c^i f^i. \end{aligned}$$

This is zero when

$$3uu'' + 2(u')^2 = 0 \quad \text{and} \quad v = 8t^{-1}u',$$

or

$$u = (k\rho+\ell)^{\frac{3}{5}}, \quad v = \tfrac{24}{5}kt^{-1}(k\rho+\ell)^{-\frac{2}{5}},$$

where k, ℓ are constants, with $kt > 0$. When $t < 0$, we must have $\rho < |k|/\ell$. □

From (7.26), the only complete self-dual Einstein 4-manifold with $t > 0$ which is spin is S^4. In this case, the metric

$$(\rho+1)^{\frac{3}{5}}\pi^*g^M + \tfrac{24}{5}t^{-1}(\rho+1)^{-\frac{2}{5}}\sum_{i=5}^{8} E^i \otimes E^i \tag{12.20}$$

on $\sigma_- S^4$ is complete, and using its invariance under the group $SO(5)$ of isometries lifted from S^4, it can be shown that there are no parallel 1-forms or 2-forms. The classification **10.7** then confirms that the holonomy group is actually *equal* to $Spin(7)$.

12.10 Corollary [BS] *The total space of the spin bundle over S^4 has a complete Ricci-flat metric with holonomy equal to $Spin(7)$.*

The above constructions are reminiscent of those of Calabi [Ca_3], and form part of a more general framework for studying Einstein metrics on total spaces of bundles pioneered by Bérard Bergery [B],[Bes]. They also generalize more familiar situations in which new metrics are built from Riemannian submersions, but a crucial feature is that the curvature of the bundle in question satisfy the Yang-Mills equations. In this context, we should have remarked earlier that the condition that a 4-dimensional *manifold* be self-dual and Einstein is equivalent to the condition that the induced connection on its associated *bundles* $\Lambda^2_- M$, $\sigma_- M$ satisfy the self-dual Yang-Mills equations [AHS]. The results are particularly effective when, like in our examples, a Lie group G acts on the total space with orbits of codimension one. In fact, a general programme might consist of determining, for each G, the set of all G-invariant cohomogeneity one Einstein metrics.

The spin bundle over $S^4 \cong \mathbb{H}P^1$ can be identified with the tautological quaternionic line bundle with fibre σ in (9.10). Its total space can then be identified with quaternionic projective plane with a point removed, and there is some analogy between (12.20) and the $Sp(2)Sp(1)$-invariant quaternionic Kähler metric on this vector bundle. The metric (12.20) is asymptotic to

$$dr^2 + r^2 ds^2, \quad r = \rho^{3/10},$$

where ds^2 is a non-standard "squashed" Einstein metric on the sphere S^7. In particular, the $Spin(7)$ metric is not asymptotically locally Euclidean like that of Calabi-Eguchi-Hanson (8.8).

Page and Pope [PP] have made a systematic study of Einstein metrics on the total spaces of bundles over quaternionic Kähler manifolds. The higher-dimensional situations are also relevant to G_2 and $Spin(7)$-metrics, given the construction of Galicki and Lawson [GL] of self-dual Einstein orbifolds. In fact, it is tempting to look for a direct reduction process for metrics with holonomy G_2 and $Spin(7)$, based on their characterization by closed 3-forms and 4-forms.

Notation index

The list includes most symbols which are introduced in the text by some form of definition and subsequently appear in another chapter. Page numbers refer to selected occurrences. Greek and other symbols are at the end.

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