Index theory and special structures on 8-manifolds

An introduction

Simon Salamon
15 July 2014

1.1 Subgroups

$$\begin{array}{ccccc} G_2 & \subset & \boxed{\mathrm{Spin}\,7} & \subset & \mathrm{Spin}\,8 \\ & & \cup & & \\ & & \mathrm{SU}(4) & & \cup \\ & & \cup & & \\ & & \mathrm{Sp}(2) & \subset & \boxed{\mathrm{Sp}(2)\mathrm{Sp}(1)} \end{array}$$

$\mathrm{Sp}(2)$	fixes a HK triple $\omega_1, \omega_2, \omega_3$
$\mathrm{Sp}(2)\mathrm{Sp}(1)$	is the stabilizer of $\omega_1^2 + \omega_2^2 + \omega_3^2 = \Omega$
$\mathrm{Spin}\,7$	is the stabilizer of $-\omega_1^2 + \omega_2^2 + \omega_3^2 = \Phi$ [BH]

1.2 Euler number

Proposition [GG]. If M^8 (compact and oriented) has a $\mathrm{Spin}\,7$ or an $\mathrm{Sp}(2)\mathrm{Sp}(1)$ structure then

$$\boxed{8\chi = 4p_2 - p_1^2}$$

Proof. For an $\mathrm{SU}(4)$ structure, $TM_c = T^{1,0} \oplus T^{0,1}$ has total Chern class

$$\begin{aligned} 1 - p_1 + p_2 &= (1 + c_2 + c_3 + c_4)(1 + c_2 - c_3 + c_4) \\ &= 1 + 2c_2 + (c_2^2 + 2c_4) \end{aligned}$$

so

$$8\varepsilon = 8c_4 = 4p_2 - p_1^2.$$

Argument extends because $\mathrm{SU}(4),\ \mathrm{Spin}\,7$, $\mathrm{Sp}(2)\mathrm{Sp}(1)$ share a maximal 3-torus!

1.3 Triality

$$\begin{array}{ll} \mathrm{Spin}\,8 & \text{acts on } \Delta = \Delta_+ \oplus \Delta_- \\ \quad\downarrow 2\!:1 & \\ \mathrm{SO}(8) & \text{acts on } T = \Lambda^1 \end{array}$$

Outer automorphisms of $\mathrm{Spin}\,8$ permute T, Δ_+, Δ_-.

Restricting to a maximal 4-torus,

$$\begin{array}{lll} \Lambda^1 & \text{has 8 weights} & \pm x_1, \pm x_2, \pm x_3, \pm x_4 \\ \Delta_+ \oplus \Delta_- & \text{has 16 weights} & \frac{1}{2}(\pm x_1 \pm x_2 \pm x_3 \pm x_4) \end{array}$$

(with an even number of like signs for Δ_+).

If $\sum x_i = 0$ then Λ^1, Δ_- have the same weights.
In fact $\Lambda^1 \cong \Delta_-$ as $\mathrm{Spin}\,7$, $\mathrm{Sp}(2)\mathrm{Sp}(1)$ modules.

In general, $\Delta \otimes \Delta = \bigoplus \Lambda^i$. Here,

$$\begin{array}{rcl} \Delta_+ \otimes \Delta_+ & \cong & \Lambda^4_+ \oplus \Lambda^2 \oplus \Lambda^0 \\ \Delta_+ \otimes \Delta_- & \cong & \Lambda^3 \oplus \Lambda^1 \end{array}$$

2.1 A hat class

The complexified tangent bundle has Chern class

$$c(TM_c) = \prod_1^4 (1 - x_i^2)$$

so $p_1 = \sum x_i^2$. Its Chern character is

$$\mathrm{ch}(TM_c) = \sum_1^4 (e^{x_i} + e^{-x_i}) = 8 + 2\sum x_i^2 + \tfrac{1}{12}\sum x_i^4$$

Similarly,

$$\mathrm{ch}(\Delta_+ - \Delta_-) = \prod_1^4 (e^{x_i/2} - e^{-x_i/2}) = \varepsilon \hat{A}(M)^{-1}$$

where $\varepsilon = x_1 x_2 x_3 x_4$ is the Euler class, and we *define*

$$\hat{A}(M) = \prod_1^4 \frac{x_i/2}{\sinh(x_i/2)} = 1 - \tfrac{1}{24}p_1 + \tfrac{1}{5760}(7p_1^2 - 4p_2) + \cdots$$

2.2 Dirac operator

This is defined as $\gamma \circ \nabla$ where m is Clifford mult:

$$\Gamma(M, \Delta_+) \xrightarrow{\not\partial} \Gamma(M, \Delta_-)$$

Given a vector bundle V with connection, it extends to an elliptic operator

$$\Gamma(M, \Delta_+ \otimes V) \xrightarrow{\not\partial_V} \Gamma(M, \Delta_- \otimes V).$$

The resulting index $\operatorname{ind}(V) = \dim\ker \not\partial - \dim \operatorname{coker} \not\partial$ depends only on the topology of V :

Theorem [AS]

$$\boxed{\operatorname{ind}(V) = \int_M \operatorname{ch}(V) \hat{A}(M)}$$

- $V = \Delta_+ - \Delta_-$ gives the 2-step de Rham complex

$$\bigoplus_{i=0}^{4} \Lambda^{2i} \xrightarrow{d+d^*} \bigoplus_{i=1}^{4} \Lambda^{2i-1}.$$

Of course, $\operatorname{ind}(V) = \chi = \sum_{i=0}^{8} (-1)^i b_i.$

2.3 Betti numbers

- $V = \mathbb{C}$ equates the index of $\partial\!\!\!/$ with

$$\boxed{\hat{A} = \hat{A}_2 = \tfrac{1}{5760}(7p_1^2 - 4p_2)}$$

- $V = \Delta_+ + \Delta_-$ gives rise to the signature operator

$$\bigoplus_{i=0}^{4^+} \Lambda^i \longrightarrow \bigoplus_{i=4^-}^{8} \Lambda^i$$

and so $\mathrm{ind}(V) = b_4^+ - b_4^- = \tau$. But

$$\mathrm{ind}(V) = \int_M \big(16 + 2p_1 + \tfrac{1}{24}(p_1^2 + 4p_2)\big)\, \hat{A}(M)$$

and

$$\boxed{\tau = \tfrac{1}{45}(7p_2 - p_1^2) = L_2}$$

Corollary. With a reduction to $\mathrm{Spin}\,7$ or $\mathrm{Sp}(2)\mathrm{Sp}(1)$,

$$\boxed{48\hat{A} = 3\tau - \chi}$$

and $24\hat{A} = -1 + b_1 - b_2 + b_3 + b^+ - 2b^-$.

2.4 Parallel spinors

- M is QK (holonomy $\subseteq \mathrm{Sp}(2)\mathrm{Sp}(1)$) with $R > 0$ so 'nearly hyperkähler'

$\Rightarrow \ \hat{A} = 0$

Also $b_3 = b^- = 0$ so $\boxed{b_4 = 1 + b_2}$

- M has holonomy equal to $\mathrm{Spin}\,7$

$\Rightarrow \ \hat{A} = 1$

Thus $\boxed{b_3 + b^+ = 25 + b_2 + 2b^-}$

- M is irreducible HK (holonomy $= \mathrm{Sp}(2)$)

$\Rightarrow \ \hat{A} = 3$

Then $\boxed{b_3 + b_4 = 46 + 10b_2 \geqslant 76}$

Beauville's have $(b_2, b_3, b_4) = (23, 0, 276), (7, 8, 108)$ [G].

3.1 HK constraint

Suppose that M^{4n} has holonomy $\mathrm{Sp}(n)$ with $\chi \neq 0$. Set

$$P(t) = \sum_{i=0}^{4n} b_i t^i.$$

Then $\chi = P(-1)$ and $P'(-1) = -2nP(-1)$. Consider

$$\log \frac{P(-1+t)}{P(-1)} = \log\left(1 - 2nt + \frac{P''(-1)}{2P(-1)} + \cdots\right) = -2nt + \tfrac{1}{2}\phi\, t^2 + \cdots$$

where $\phi + 4n^2 = \frac{P''(-1)}{P(-1)}$. By construction,

$$\phi(M \times N) = \phi(M) + \phi(N)$$

is additive.

Theorem [S]. Any cpt HK manifold M^{4n} has $\phi = -5n/3$. Equivalently

$$n\chi = 6 \sum_{i=0}^{2n-1} (-1)^i (2n-i)^2 b_i$$

and as a corollary, $24 \mid (n\chi)$.

$n = 1 \;\Rightarrow\; 4b_1 + b_2 = 22$
$n = 2 \;\Rightarrow\; 25b_1 - 10b_2 + b_3 + b_4 = 46.$

ϕ plays a role in the theory of symmetric holonomy.

3.2 QK topology

By analogy to $\mathrm{Spin}\,8/\mathrm{Spin}\,7 \cong S^7$,

$$\frac{\mathrm{Spin}\,8}{\mathrm{Sp}(2)\mathrm{Sp}(1)} \cong \frac{\mathrm{SO}(8)}{\mathrm{SO}(5)\times\mathrm{SO}(3)} = \mathbb{G}\mathrm{r}_3(\mathbb{R}^8)$$

Given an $\mathrm{Sp}(2)\mathrm{Sp}(1)$ structure,

$$TM_c \;\cong\; E\otimes H \;\cong\; \Delta_-$$
$$\Delta_+ \;\cong\; \Lambda^2_0 E \oplus S^2 H$$

where $S^2H \cong \langle I, J, K\rangle_c$. We have

$$\Delta_+ - \Delta_- = \Lambda^2_0 E - E\otimes H + S^2 H = \Lambda^2_0(E-H).$$

Proposition. Relative to $\mathrm{Sp}(n)\mathrm{Sp}(1)$,

$$V = \Delta_+ - \Delta_- \cong \Lambda^n_0(E-H).$$

This explains why $\mathrm{ch}(V) \in H^{4n}(M,\mathbb{R})$. Similar techniques can be used in other situations to prove, e.g. $\mathcal{M}_g$ inside $\mathcal{F}_g \to \mathbb{G}\mathrm{r}_4(2g{+}2)$ has $T\mathcal{M}_g = Q\otimes W - \psi^2 Q$ and $p_1^g = 0$ [K].

3.3 Isometry groups

Over $M = \mathbb{HP}^2$, H is the tautological line bundle. In general,

$$h = -4c_2(H) \in H^4(M, \mathbb{Z})$$

represents the class generated by Ω, and $h^2 \in \mathbb{N}$.

Proposition. Suppose M^8 is QK with $R > 0$. Then

$$\mathrm{ind}(S^2H) = 1, \ \ \mathrm{ind}(TM) = -1 - b_2, \ \ \mathrm{ind}(\Lambda^2_0 E) = 2b_2 + 1$$

The corresponding modules are trivial representations of the isometry group G. On the other hand,

$$\mathrm{ind}(S^4H) = \dim G = 5 + h^2 \ \geqslant \ 6.$$

By twistor/Mori theory, one knows that $b_2 > 0$ implies $M \cong \mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$. So we can assume $b_2 = 0$ and $b_4 = 1$. Then

$$\dim G = 21, \ 14, \ 9, \ 6.$$

The first two cases are $\mathbb{HP}^2$ and $\mathrm{G}_2/SO(4)$, the other two can be eliminated [PS].

3.4 Rigid operators

When an isometry group S^1 or G acts on M, the indices become virtual G modules. Operators like $\partial\!\!\!/ \otimes \Delta_+$ that involve Betti numbers will be *rigid*, meaning that the indices are sums of trivial modules.

Theorem [AH]. If a compact spin manifold admits a non-trivial S^1 action then $\hat{A} = 0$. (cf. Spin 7)

Define a sequence of virtual vector bundles R'_i by

$$R'(q) = \sum_{i=0}^{\infty} q^k R'_k = \bigotimes_{i=1}^{\infty} \Lambda(q^{2i-1})/\Lambda(-q^{2i}).$$

Explicitly,
$$\begin{aligned} R'_0 &= \mathbb{C} \\ R'_1 &= TM = \Lambda^1 \quad \text{(Rarita-Schwinger)} \\ R'_2 &= \Lambda^2 \oplus \Lambda^1 \\ R'_3 &= \Lambda^3 + \Lambda^2 + \mathsf{S}^2 + \Lambda^1 \\ R'_4 &= 2\Lambda^1 + \Lambda^2 + \Lambda^3 + \Lambda^4 + 2\mathsf{S}^2 + V_{\mathrm{sw}} \end{aligned}$$

Theorem [W, BT]. If M^{2n} is a compact spin manifold then $\mathrm{ind}(R'_k)$ is rigid for each k.

Strategy: $\mathrm{ind}(R'(q))$ is a mero function on $\mathbb{C}/\langle 1, e^{2\pi i q}\rangle$.

4.1 Fernández example

Theorem [CF]. There are 12 nilpotent Lie algebras $\mathfrak{g}$ that admit left-invariant G_2 structures with $d\varphi = 0$.

They all give rise to compact nilmanifolds $N = \Gamma\backslash G$ but with $\pi_1 = \Gamma$ infinite and $p_1 = 0$! An easy one is

$$\mathfrak{g}^* = \langle e^1, \dots, e^7 \rangle = \mathfrak{g}_5 \oplus \mathbb{R}^2$$

with $de^4 = e^{12}$ and $de^5 = e^{13}$. The closed 3-form is

$$(45 - 67)1 + (46 - 75)2 + (47 - 56)3 + 123.$$

The cohomology ring of N is isomorphic to that of the DGA $(\bigoplus \Lambda^i \mathfrak{g}^*, d)$ [N], which:
(i) is freely generated by $e^1, \dots, e^7$,
(ii) has a nilpotent property $de^k \in \bigwedge^2 \langle e^1, \dots, e^{k-1} \rangle$.
This means that it is a *minimal model* for the de Rham algebra (over $\mathbb{R}$). It has a non-zero Massey product

$$\langle [e^2], [e^1], [e^3] \rangle = [-e^{43} + e^{25}] \in H^2(N, \mathbb{R}) / \langle [e^3], [e^2] \rangle.$$

4.2 Formality

If M is simply connected or 'nilpotent', its minimal model determines $\pi_*(M)\otimes\mathbb{Q}$.

A manifold is *formal* if there is a morphism of its mimimal model to $(H^*(M,\mathbb{R}),d=0)$ inducing an isomorphism on cohomology (so the latter determines rational homotopy). All Massey products must vanish.

• Spheres are formal: e.g. the minimal model for S^{2n} is $\mathsf{S}(x_{2n})\otimes\Lambda(y_{4n-1})$ with $dy=x^2$. Indeed, $\pi_i(S^{2n})\otimes\mathbb{Q}$ is non-trivial iff $i=0,2n,4n-1$.

• Compact symmetric spaces are formal: cohomology is represented by parallel forms, so Massey products vanish.

• Compact Kähler manifolds are formal, thanks to the $\partial\overline{\partial}$ lemma and Chern's theorem [DGMS].

• A nilmanifold is formal only if it is a torus [H], so nilmanifolds can't be Kähler even though many admit both complex and symplectic structures.

• Positive QK manifolds are formal, because they have Kähler twistor spaces [A].

4.3 Low dimensions

• Any simply-connected (compact oriented) 6-manifold is formal. Any k-connected manifold M^n with $n \leqslant 4k+2$ is formal [M].

• Any M^7 or M^8 with $b_2 \leqslant 1$ is formal [C].

• There exist simply-connected symplectic manifolds M^{2n} that are not formal for all $n \geqslant 4$.

• Which of the known manifolds with holonomy G_2 or $\mathrm{Spin}\,7$ are formal? Many have vanishing Massey products since $[\alpha] \cup [\beta] = 0$ implies that $\alpha \wedge \beta = 0$ as forms.

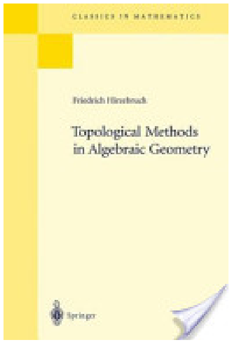
Why is the cohomology ring of a manifold with special holonomy compatible with index theory constraints, like $b_3 + b^+ = 25 + b_2 + 2b^-$ for $\mathrm{Spin}\,7$?

What about the topology of compact 8-manifolds with holonomy $\mathrm{Sp}(2)\mathrm{Sp}(1)$ and $R < 0$?

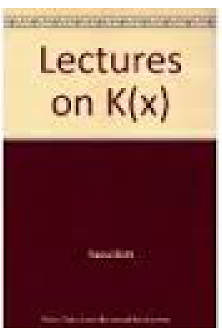
5.1 References

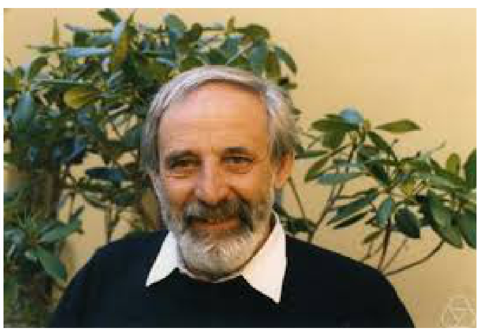
[A]	Amman
[AH]	Atiyan and Hirzebruch
[AS]	Atiyan and Singer
[BH]	Bryant and Harvey
[BT]	Bott and Taubes
[C]	Cavalcanti
[CF]	Conti and Fernández
[DGMS]	Deligne, Griffiths, Morgan and Sullivan
[G]	Guan
[GG]	Gray and Green
[H]	Hasegawa
[K]	Kirwan
[M]	Miller
[N]	Nomizu
[PS]	Poon and Salamon
[S]	Salamon
[W]	Witten

5.2 Bibliography

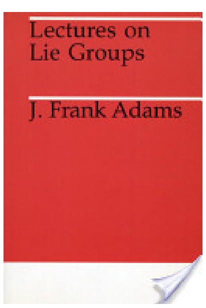



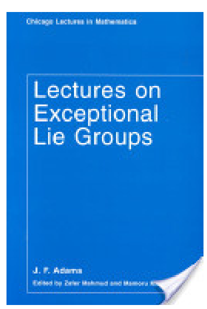
Topological Methods in Alg Geometry, F. Hirzebruch 1927–2012





Lectures on K(X), by R. Bott 1923-2005



Lectures on Lie groups, by J.F. Adams 1930-1989