1
Hermitian Geometry

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Introduction

These notes are based on graduate courses given by the author in 1998 and 1999. The main idea was to introduce a number of aspects of the theory of complex and symplectic structures that depend on the existence of a compatible Riemannian metric. The present notes deal mostly with the complex case, and are designed to introduce a selection of topics and examples in differential geometry, accessible to anyone with an acquaintance of the definitions of smooth manifolds and vector bundles.

One basic problem is, given a Riemannian manifold (M, g), to determine whether there exists an orthogonal complex structure J on M, and to classify all such J. A second problem is, given (M, g), to determine whether there exists an orthogonal almost-complex structure for which the corresponding 2-form ω is closed. The first problem is more tractible in the sense that there is an easily identifiable curvature obstruction to the existence of J, and this obstruction can be interpreted in terms of an auxiliary almost-complex structure on the twistor space of M. This leads to a reasonably complete resolution of the problem in the first non-trivial case, that of four real dimensions. The theory has both local and global aspects that are illustrated in Pontecorvo's classification [89] of bihermitian anti-self-dual 4-manifolds. Much less in known in higher dimensions, and some of the basic classification questions concerning orthogonal complex structures on Riemannian 6-manifolds remain unanswered.

The second problem encompasses the so-called Goldberg conjecture. If one believes this, there do not exist compact Einstein almost-Kähler manifolds for which the metric is not actually Kähler [50]. A thorough investigation of the associated geometry requires an exhaustive analysis of high order curvature jets, and we explain this briefly in the 4-dimensional case. Although we shall pursue the resulting theory elsewhere, readers interested in almost-Kähler manifolds may find some sections of these notes, such as those describing curvature, relevant to this subject.

Complex and almost-complex manifolds are introduced in the first section, and there is a discussion of integrability conditions in terms of exterior forms and their decomposition into types. As an application, we discuss left invariant complex structures on Lie groups, which provide a casual source of examples throughout the notes. The second section is devoted to a study of almost-Hermitian manifolds. It incorporates a brief description of Hodge theory for the $\overline{\partial}$ operator, in order to explain what is special in the Kähler case, though this topic is covered in detail in many standard texts. We also study the space of almost-complex structures compatible with the standard inner product on $\mathbb{R}^{2n}$, and explain how this leads to the definition of twistor space. Attention is given to the case of four dimensions, and examples arising from self-duality, in preparation for more detailed treatment.

The third section introduces the theory of connections on vector bundles by way of covariant differentiation. Basic results for connections compatible with either a Riemannian or symplectic structure exhibit a certain amount of duality involving symmetric versus skew-symmetric tensors. Analogous identities are derived for the complex case, and these lead to the obstruction mentioned above. The subsequent analysis of the Riemann curvature tensor of an almost-Hermitian manifold is partially lifted from [38], and leads straight on to a treatment of the four-dimensional case in the last section. This is followed by a discussion of special Kähler metrics that arise from the study of algebraically integrable systems. The notes conclude with the re-interpretation of a 4-dimensional hyperkähler example constructed earlier from a solvable Lie group.

One aim of the original courses was to introduce the audience to the material underlying some current research papers. The author had partly in mind Apostolov-Grantcharov-Gauduchon [7], Freed [42], Hitchin [59], Poon-Pedersen [87], Gelfand-Retakh-Shubin [47], and there are some direct references to these. The author had useful conversations with the majority of these authors, and is grateful for their help. Even when coverage of a particular topic is scant, we have tried to supply some of the basic references. In this way, readers will know where to look for a more comprehensive and effective treatment.

The author's first exposure to differential geometry took place during one of Brian Steer's undergraduate tutorials, in which some scheduled topic was replaced by a discussion of connections on manifolds. This was after ground had been prepared by the inclusion of books such as [81] in a vacation reading list. I am for ever grateful for Brian's guidance and the influence he had in my choice of research.

Contents

1 Complex manifolds

Holomorphic functions

Let U be an open set of $\mathbb{C}$, and $f: U \to \mathbb{C}$ a continuously differentiable mapping. We may write

$$f = u + iv = u(x,y) + iv(x,y), \tag{1.1}$$

where $x + iy$ and $u + iv$ are complex coordinates on the domain and target respectively. The function f is said to be *holomorphic* if the resulting differential $df = f_*: \mathbb{R}^2 \to \mathbb{R}^2$ is complex linear. This means that the Jacobian matrix of partial derivatives commutes with multiplication by i, so that

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

and one obtains the Cauchy-Riemann equations

$$\begin{cases} u_x &= v_y, \\ v_x &= -u_y. \end{cases}$$

The differential of a holomorphic function then has the form

$$df = u_x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + v_x \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This corresponds to multiplication by the complex scalar $u_x + iv_x$, which coincides with the complex limit

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

Now let $z = x + iy$ and $\overline{z} = x - iy$, so that the mapping (1.1) may be regarded as a function $f(z, \overline{z})$ of the variables $z, \overline{z}$. Define complex-valued 1-forms

$$dz = dx + i dy, \quad d\overline{z} = dx - i dy,$$

which are dual to the tangent vectors

$$\frac{\partial}{\partial z} = \tfrac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial \overline{z}} = \tfrac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

Then f is holomorphic if and only if

$$\frac{\partial f}{\partial \overline{z}} = 0,$$

which is the case whenever f may be expressed explicitly in terms of z alone. Indeed, Cauchy's theorem implies that f is holomorphic if and only if it is complex analytic, i.e. if for every $z_0 \in U$ it has a convergent power series expansion in $z - z_0$ valid on some disk around z_0.

Now suppose that U is an open set of $\mathbb{C}^m$ and that $F\colon U \to \mathbb{C}^n$ is a continuously differentiable mapping. It is convenient to identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$ by means of the association

$$(z^1,\dots,z^m) \ \leftrightarrow\ (x^1,\dots,x^m,y^1,\dots,y^m),$$

where $z^r = x^r + iy^r$. Then F is *holomorphic* if and only if

$$dF\circ\begin{pmatrix} 0 & -I\\ I & 0\end{pmatrix} = \begin{pmatrix} 0 & -I\\ I & 0\end{pmatrix}\circ dF,$$

where I denotes an identity matrix of appropriate size. Equivalently,

$$\frac{\partial F^j}{\partial \overline{z}^k} = 0, \quad j=1,\dots,n,\ k=1,\dots,m. \tag{1.2}$$

A complex manifold is a smooth manifold M, equipped with a smooth atlas with the additional property that its transition functions are holomorphic wherever defined. Thus, a complex manifold has an open covering $\{U_\alpha\}$ and local diffeomorphisms

$$\phi_\alpha\colon U_\alpha \to \phi(U_\alpha) \subseteq \mathbb{C}^m$$

with the property that $\phi_\alpha\circ\phi_\beta^{-1}$ is holomorphic whenever $U_\alpha\cap U_\beta\neq\emptyset$. If x is a point of U_α and $\phi_\alpha = (z^1,\dots,z^m)$ then $z^1,\dots,z^m$ are called *local holomorphic coordinates* near x. The complex structure is deemed to depend only on the atlas up to the usual notion of equivalence, whereby two atlases are equivalent if their mutual transition functons are holomorphic.

Example 1.3 Let Γ denote the additive subgroup of $\mathbb{C}^m$ generated by a set of $2m$ vectors, linearly independent over $\mathbb{R}$. Then $M=\mathbb{C}^m/\Gamma$ is diffeomorphic to the real torus $\mathbb{R}^{2m}/\mathbb{Z}^{2m} = (\mathbb{R}/\mathbb{Z})^{2m} \cong (S^1)^{2m}$. Let $\pi\colon\mathbb{C}^m \to M$ be the projection. An atlas is constructed from pairs (U_α,ϕ_α) where $\pi\circ\phi_\alpha$ is the identity on U_α, and the transition functions are translations by elements of $\mathbb{Z}^m$ in $\mathbb{C}^m$.

Fix a point of M, and let T denote real the tangent space to M at that point, $T_{\mathbb{C}}$ its complexification $T\oplus iT$. The endomorphism J of T defined by

$$(d\phi_a)^{-1}\circ\begin{pmatrix} 0 & -I\\ I & 0\end{pmatrix}\circ d\phi_a$$

is independent of the choice of chart, since $d(\phi_a^{-1}\circ\phi_b) = d\phi_\alpha^{-1}\circ d\phi_b$ is complex-linear. The same is true of the subspace $T^{1,0}$ of $T_{\mathbb{C}}$ generated by the tangent vectors $\partial/\partial z^1,\dots,\partial/\partial z^m$, since this coincides with the i eigenspace $\{X-iJX : X\in T\}$, and

$$\frac{\partial}{\partial z^r} = \sum_{s=1}^m \frac{\partial w^s}{\partial z^r}\frac{\partial}{\partial w^s}.$$

Setting

$$T^{0,1} = \overline{T^{1,0}} = \langle \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_m} \rangle$$

gives a decomposition

$$T_{\mathbb{C}} = T^{1,0} \oplus T^{0,1}.$$

There is a dual decomposition

$$T^*_{\mathbb{C}} = \Lambda^{1,0} \oplus \Lambda^{0,1}, \tag{1.4}$$

where $\Lambda^{1,0}$ is the annihilator of $T^{0,1}$ spanned by $dz^1, \ldots, dz^m$. Varying from point to point, we may equally well regard $T^*_{\mathbb{C}}$ as the complexified cotangent *bundle*, and (1.4) as a decomposition of it into conjugate subbundles. The endomorphism J acts on T^* by the rule $(J\alpha)(v) = \alpha(Jv)$ and this implies that

$$J\,dz^r = i\,dz^r, \quad \text{or} \quad J\,dx^r = -dy^r.$$

Thus a function $x + iy$ with real and imaginary components x, y is holomorphic if and only if $dx - Jdy = 0$.

A *holomorphic function* on a complex manifold M is a mapping $M \to \mathbb{C}$ such that $f \circ \phi$ is holomorphic on U for any chart (U, ϕ). If $f \circ \phi$ is allowed to have poles at isolated points then f is merely *meromorphic*. On a *compact* complex manifold M with charts $\phi_\alpha : U_\alpha \to \mathbb{C}^m$, any holomorphic function is constant, whereas the field of meromorphic functions can be used to define the *algebraic dimension* $a(M)$, which satisfies $a(M) \leqslant m$.

Holomorphic mappings between complex manifolds are defined by reference to charts and the resulting condition (1.2), with the corresponding generalization to meromorphic case. Two complex manifolds M, N are called *biholomorphic* if there exists a bijective holomorphic mapping $f : M \to N$; in this case f^{-1} is automatically holomorphic. Biholomorphism is the natural equivalence relation between complex manifolds, although other notions are important in the realm of algebraic geometry. For example, if there exist open sets $U \subseteq M$ and $V \subseteq N$ and a biholomorphic map $f : U \to V$, then M and N are *birational*. For surfaces this implies $a(M) = a(N)$.

Exercises 1.5 (i) Prove that if M is a compact complex manifold, any holomorphic mapping from M to the complex numbers $\mathbb{C}$ is necessarily constant. Give an example of a complex manifold for which there is no holomorphic mapping $\mathbb{C} \to M$ (it may help to know that such a complex manifold is called *hyperbolic*).

(ii) Let Γ be a discrete group, and M a complex manifold. Say what is meant by a *holomorphic* action of Γ on M, and give sufficient conditions for the set M/Γ of cosets to be a complex manifold for which the projecton $\pi : M \to M/\Gamma$ is holomorphic.

Examples and special classes

The basic compact example of a complex manifold is the complex projective space

$$\mathbb{CP}^m = \{\text{1-dimensional subspaces of } \mathbb{C}^{m+1}\} = \frac{\mathbb{C}^{m+1} \setminus \{0\}}{\mathbb{C}^*}.$$

A point of $\mathbb{CP}^m$ is denoted $[Z^0, Z^1, \dots, Z^m]$, and represents the span of a non-zero vector in $\mathbb{C}^{m+1}$. An open set U_α is defined by the condition $Z^\alpha \neq 0$, for each $\alpha = 0, 1, \dots, n$, and holomorphic coordinates are given by Z^r/Z^α on U_α. For example, fix $\alpha = 0$ and set $z^r = Z^r/Z^0$. Then transition functions on $U_0 \cap U_1$ are given by

$$(z^1, \dots, z^m) \longmapsto (\frac{1}{z^1}, \frac{z^2}{z^1}, \dots, \frac{z^m}{z^1}), \quad z^1 \neq 0,$$

and are clearly holomorphic.

Definition 1.6 *A compact complex manifold M is projective if there exists a holomorphic embedding of M into $\mathbb{CP}^n$ for some n.*

Projective manifolds are most easily defined as the zeros of homogeneous polynomials in $\mathbb{CP}^n$. For example, if F is a homogeneous polynomial of degree d in $Z^0 = X, Z^1 = Y, Z^2 = Z$ then

$$M_F = \{[X, Y, Z] \in \mathbb{CP}^2 : F(X, Y, Z) = 0\}$$

is a 1-dimensional complex manifold of the projective plane $\mathbb{CP}^2$ provided $F_X = F_Y = F_Z = 0$ implies $X = Y = Z = 0$. Note that the vanishing of the partial derivative of F at a point implies the vanishing of F at that point, by Euler's formula. A complex manifold of complex dimension 1 is called a *Riemann surface*, and is necessarily orientable.

Example 1.7 To be more specific, consider the hypersurface M_F of degree d defined by the equation $X^d + Y^d + Z^d = 0$. The projection

$$\begin{aligned} M_F &\to \mathbb{CP}^1 \\ (X, Y, Z) &\longmapsto (Y, Z) \end{aligned}$$

is a 'branched $d : 1$ covering': each fibre has d points unless $Y^d + Z^d = 0$, an equation that has d solutions in $\mathbb{CP}^1$. The resulting topology can be derived from the basic set-theoretic properties of the Euler characteristic χ, that imply that

$$\chi(M_F) = d\,\chi(\mathbb{CP}^1 \setminus d \text{ points}) + d = d(2 - d) + d.$$

Thus the genus of the resulting compact oriented surface satisfies $2 - 2g = -d^2 + 3d$, and

$$g = \frac{(d-1)(d-2)}{2}. \tag{1.8}$$

The degree-genus formula (1.8) is always valid for a smooth plane curve of degree d (see [66] for a thorough treatment). The cases $d = 1, 2$ are elementary. If $d = 3$ then $g = 1$, and a smooth cubic curve in $\mathbb{CP}^2$ is topologically a torus. It is known that any such cubic is projectively equivalent to

$$C_\lambda : \quad Y^2 Z = X(X - Z)(X - \lambda Z), \quad \lambda \notin \{0, 1\},$$

and that C_λ, C_μ are equivalent iff

$$\mu \in \left\{ \lambda, \frac{1}{\lambda}, 1 - \lambda, \frac{1}{1 - \lambda}, \frac{\lambda}{\lambda - 1}, \frac{\lambda - 1}{\lambda} \right\},$$

in which case

$$j = \frac{(\mu^2 - \mu + 1)^3}{\mu^2(\mu - 1)^2}$$

has a unique value. The manifold C_μ is biholomorphic to the quotient group $\mathbb{C}^2/\langle 1, \tau \rangle$, where τ belongs to the upper half plane. The special case $\tau = i$ corresponds to $j = 27/4$, but in general the mapping $\tau \longmapsto j$ involves non-trivial function theory.

If $d = 4$ then $g = 3$, so no smooth curve in $\mathbb{CP}^2$ has genus 2. But that does not of course mean to say that a torus with 2 holes does not admit a complex structure. Indeed, any oriented surface can be made into a complex manifold by first embedding it in $\mathbb{R}^3$. The classical isothermal coordinates theorem asserts that, given a surface in $\mathbb{R}^3$, there exist local coordinates x, y such that the first fundamental form equals $f(dx^2 + dy^2)$ for some positive function f. It then suffices to define $J\frac{\partial}{\partial x} = \frac{\partial}{\partial y}$; transition functions will be automatically holomorphic.

Any Riemann surface of genus at least 2 is biholomorphic to Δ/Γ, where Δ denotes the open unit ball in $\mathbb{C}$, and Γ is a discrete group acting holomorphically. Many important classes of complex manifolds may be defined by taking discrete quotients of an open set of $\mathbb{C}^m$.

Example 1.9 We have already mentioned complex tori $\mathbb{C}^m/\Gamma$, in which the lattice Γ is a subgroup of the abelian group $(\mathbb{C}^m, +)$. The *Kodaira surface* S is obtained in a not dissimilar manner by regarding $\mathbb{C}^2$ as the real nilpotent group of matrices

$$\begin{pmatrix} 1 & x & u & t \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

under multiplication. If Γ denotes the subgroup for which x, y, u, t are integers then S is the quotient $\Gamma\backslash\mathbb{C}^2$ of $\mathbb{C}^2$ by left-multiplication of Γ. This action leaves invariant the 1-forms

$$dx, \quad dy, \quad du - x\,dy, \quad dt,$$

and therefore the closed 2-form $\eta = (dx + idy) \wedge (du - xdy + idt)$ which defines a holomorphic symplectic structure on S [69]. Foundations of a general theory of such compact quotients of nilpotent Lie groups are established in [79]. To cite just two applications of Kodaira surfaces, refer to [46; 55].

Other complex surfaces defined as discrete quotients, include Hopf qand Inoue surfaces [14]. A *Hopf surface* is a compact complex surface whose universal covering is biholomorphic to $\mathbb{C}^2 \setminus \{0\}$, and is called *primary* if $\pi_1 \cong \mathbb{Z}$. Any primary Hopf surface is homeomorphic to $S^1 \times S^3$.

Example 1.10 Let Γ denote the infinite cyclic group generated by a non-zero complex number λ with $|\lambda| \neq 1$. Then $M_\lambda = (\mathbb{C}^m \setminus \{0\})/\langle\lambda\rangle$ is a complex manifold diffeomorphic to $S^1 \times S^{2m-1}$. This is an example of a Calabi-Eckmann structure that exists on the product of any two odd-dimensional spheres [30]. When $m = 2$, the complex manifold M_λ is a primary Hopf surface.

An almost-complex structure on a real $2m$-dimensional vector space $T \cong \mathbb{R}^{2m}$ is a linear mapping $J\colon T \to T$ with $J^2 = -\mathbb{1}$. An almost-complex structure on a differentiable manifold is the assignment of an almost-complex structure J on each tangent space that varies smoothly. We have seen that any complex manifold is naturally equipped with such a tensor. Conversely, an almost-complex structure J is said to be *integrable* if M has the structure of a complex manifold with local coordinates $\{z^r\}$ for which $J\,dz^r = i\,dz^r$. This means that the almost-complex structure has locally the standard form

$$J = \sum_{r=1}^{n} \Big(\frac{\partial}{\partial y^r} \otimes dx^r - \frac{\partial}{\partial x^r} \otimes dy^r \Big). \tag{1.11}$$

In general this will not be true, and we shall refer to the pair (M, J) as an almost-complex manifold.

Definition 1.12 *A hypercomplex manifold is a smooth manifold equipped with a triple I_1, I_2, I_3 of integrable complex structures satisfying*

$$I_1 I_2 = I_3 = -I_2 I_1. \tag{1.13}$$

It is easy to see that such a manifold must have real dimension $4k$ for some $k \geqslant 1$. Less obvious is the fact that the integrability of just I_1 and I_2 implies (in the presence of (1.13)) that of I_3. Thus, a hypercomplex manifold can be defined by the existence of an anti-commuting pair of complex structures. It follows that $aI_1 + bI_2 + cI_3$ is a complex structure for any $(a, b, c) \in S^2$.

Exercises 1.14 (i) Show that the set $\mathbb{F}_n$ of full flags in $\mathbb{C}^n$ (that is, sequences $\{0\} \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$ of subspaces with $\dim V_i = i$) can be given the structure of a complex manifold. The choice of an appropriate line bundle determines a holomorphic embedding $\mathbb{F}_3 \to \mathbb{CP}^7$.

(ii) Show that if $m=2k$ and $\lambda\in\mathbb{R}$ in Example 1.10 then M is hypercomplex. Explain why certain Hopf surfaces admit two distinct families of hypercomplex structures [45].

(iii) Let $m>1$, and suppose that $M=\mathbb{C}^m/\Gamma$ is a complex torus as in Example 1.3. Find sufficient conditions on Γ so that M is projective. In this case M is called an *abelian variety.*

Use of differential forms

On any almost-complex manifold, we may extend (1.4) by defining

$$\textstyle\bigwedge^k T^*_{\mathbb{C}} = \bigoplus_{p+q=k} \Lambda^{p,q}, \tag{1.15}$$

where

$$\Lambda^{p,q} \cong \textstyle\bigwedge^p(\Lambda^{1,0})\otimes\bigwedge^q(\Lambda^{0,1}).$$

These summands represent either vector bundles or the vector spaces corresponding to the fibres of these bundles at a given (possibly unspecified) point, depending on the context. On a complex manifold, $\Lambda^{p,q}$ is spanned by elements $dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\overline{z}^{j_1}\wedge\cdots\wedge d\overline{z}^{j_q}$, but it is still well defined in the presence of an almost-complex structure. To avoid complexifying spaces unncecessarily, we shall make occasional use of the notation

$$[\![\Lambda^{p,q}]\!]\quad(p\neq q),\qquad [\Lambda^{p,p}] \tag{1.16}$$

of [92] for the real subspaces of $\Lambda^k T^*$ of dimension $2pq$ and p^2, with complexifications $\Lambda^{p,q}\oplus\Lambda^{q,p}$ and $\Lambda^{p,p}$ respectively.

The *Nijenhuis tensor* N of J is defined by

$$N(X,Y)=[JX,JY]-[X,Y]-J[JX,Y]-J[X,JY], \tag{1.17}$$

and

$$\begin{aligned} -N(X,Y) &= \Re\mathrm{e}\Big([X-iJX,Y-iJY]+iJ[X-iJX,Y-iJY]\Big)\\ &= 8\,\Re\mathrm{e}\Big([X^{1,0},Y^{1,0}]^{0,1}\Big). \end{aligned} \tag{1.18}$$

Lemma 1.19 *The following are equivalent:*
(i) $d(\Gamma(\Lambda^{1,0}))\subseteq\Gamma(\Lambda^{2,0}\oplus\Lambda^{1,1})$*;*
(ii) $\Gamma(T^{1,0})$ *is closed under bracket of vector fields;*
(iii) $N(X,Y)=0$ *for all vector fields* X,Y*.*

To see the equivalence of (i) and (ii), let $\alpha\in\Gamma(\Lambda^{1,0})$, and use the formula

$$2d\overline{\alpha}(A,B)=A(\overline{\alpha}B)-B(\overline{\alpha}A)-\overline{\alpha}[A,B]=-\overline{\alpha}[A,B],\quad A,B\in\Gamma(T^{1,0}).$$

The equivalence of (ii) and (iii) follows from (1.18). $\square$

The composition

$$\Gamma(\Lambda^{1,0}) \xrightarrow{d} \Gamma(\textstyle\bigwedge^2 T^*_{\mathbb{C}}) \to \Gamma(\Lambda^{0,2})$$

is an element of

$$\mathrm{Hom}(\Lambda^{1,0},\Lambda^{0,2}) \cong \Lambda^{0,2}\otimes(\Lambda^{1,0})^* \subset \textstyle\bigwedge^2 T^*_{\mathbb{C}}\otimes T_{\mathbb{C}}, \tag{1.20}$$

since any differentiation is cancelled out by the projection. It follows that the value of $N(X,Y)$ at a point p depends only on the values of X,Y at p, and is essentially the real part of the element in (1.20). It is clear that the Nijenhius tensor N vanishes on a complex manifold. The converse is a deep result of [83] that took some years to prove after it was first conjectured.

Theorem 1.21 *An almost-complex structure J is integrable if N is identically zero.*

Example 1.22 An almost-complex structure J can equally well be defined by its action on each *co*tangent space T^*. Define one on $\mathbb{R}^4$ with coordinates (x,y,u,t) by setting

$$J(dx) = -dy, \quad J(dv) = -du + xdy,$$

so that the space of $(1,0)$-forms is generated by $\alpha^1 = dx + idy$ and $\alpha^2 = dt + i(du - xdy)$. Since $d\alpha^2$ is a multiple of $\alpha^1\wedge\overline{\alpha}^1$, Theorem 1.21 implies that there must exist local holomorphic coordinates z^1, z^2. In fact, we may take

$$z^1 = x + iy, \quad z^2 = t + iu - ixy + \tfrac{1}{2}y^2,$$

since $dz^2 = \alpha^2 - iy\alpha^1$. This example is relevant to Example 1.9, since J passes to the compact quotient S.

From now on, we shall denote the space $\Gamma(\Lambda^{p,q})$ of smooth forms of type (p,q) (that is, smooth sections of the vector bundle with fibre $\Lambda^{p,q}$) by $\Omega^{p,q}$. Now suppose that M is a complex manifold. Then we may write $d = \partial + \overline{\partial}$, where

$$\partial\colon \Omega^{p,q} \to \Omega^{p+1,q}, \quad \overline{\partial}\colon \Omega^{p,q} \to \Omega^{p,q+1}.$$

Since $d^2 = 0$ we get

$$\partial^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0, \quad \overline{\partial}^2 = 0,$$

and this gives $\bigoplus\limits_{p,q}\Omega^{p,q}$ the structure of a bigraded bidifferential algebra [82].

It is convenient to use the language of spectral sequences, and we set $E_0^{p,q} = \Omega^{p,q}$ and $d_0 = \overline{\partial}\colon E_0^{p,q} \to E_0^{p,q+1}$. The Dolbeault cohomology groups of M are given by

$$H^{p,q}_{\overline{\partial}} = E_1^{p,q} = \frac{\ker(\overline{\partial}|\Omega^{p,q})}{\overline{\partial}(\Omega^{p,q-1})}, \tag{1.23}$$

and there is a linear mapping $d_1\colon E_1^{p,q} \to E_1^{p+1,q}$ induced from ∂.

The rules of a spectral sequence decree that, given

$$d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}, \quad {d_r}^2 = 0,$$

spaces $E_{r+1}^{p,q}$ are defined as the cohomology groups of the complex $(E_r^{p,q}, d_r)$. In the present case, successive differentials are obtained by diagram chasing, and d_r will vanish for $r > n+1$, so we may write $E_\infty^{p,q} = E_{n+2}^{p,q}$. Frölicher's theorem asserts that the spectral sequence converges to the deRham cohomology of M, which means that

$$H^k(M,\mathbb{R}) = \bigoplus_{p+q=k} E_\infty^{p,q},$$

in a way compatible with a natural filtration on $H^k(M,\mathbb{R})$.

To illustrate this, suppose that M has complex dimension $n=3$. We obtain six non-zero instances of d_2. For example, to define $d_2\colon E_2^{1,2} \to E_2^{3,1}$, suppose that $[[x]] \in E_2^{1,2}$ with $x \in E_0^{1,2}$. Then $\partial x = \overline{\partial} y$ for some $y \in E_0^{2,1}$, and we set $\alpha([[x]]) = [[\partial y]]$. There are only two non-zero instances of d_3, mapping $E_3^{0,2} \to E_3^{3,0}$ and $E_3^{0,3} \to E_3^{3,1}$.

$$\begin{tikzcd}
E_2^{0,3} \arrow[drr] & E_2^{1,3} \arrow[drr] & E_2^{2,3} & E_2^{3,3} \\
E_2^{0,2} \arrow[drr] & E_2^{1,2} \arrow[drr] & E_2^{2,2} & E_2^{3,2} \\
E_2^{0,1} \arrow[drr] & E_2^{1,1} \arrow[drr] & E_2^{2,1} & E_2^{3,1} \\
E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & E_2^{3,0}
\end{tikzcd} \tag{1.24}$$

The conclusion is that there is an isomorphism

$$\begin{aligned} H^3(M,\mathbb{R}) \cong (\ker d_3 \text{ in } E_3^{0,3}) \oplus (\ker d_2 \text{ in } E_2^{1,2}) \\ \oplus (\operatorname{coker} d_2 \text{ in } E_2^{2,1}) \oplus (\operatorname{coker} d_3 \text{ in } E_3^{3,0}). \end{aligned}$$

On a compact manifold for which (1.23) are finite-dimensional, one deduces that

$$b_k \leqslant \sum_{p+q=k} h^{p,q}, \tag{1.25}$$

since $\dim E_{r+1}^{p,q} \leqslant \dim E_r^{p,q}$ for all r. It is known that if M is compact complex surface (i.e., $n = 2$) then all the differentials d_2 vanish so there is equality [14]. The wealth of homogeneous spaces in [108] provide test cases for $n = 3$, though explicit examples for which d_r does not vanish for $r \geqslant 3$ are rare (in the literature, but probably not in real life). Instances arising from invariant complex structures on compact Lie groups are described in [88].

Exercises 1.26 (i) Discover the definition of the *sheaf* $\mathcal{O}$ of germs of local holomorphic functions on a complex manifold M, and that of the Čech cohomology groups of $\mathcal{O}$.

(ii) Find out how to prove the following, at least for $q = 1$. Let $m \in M$, and suppose that $\alpha \in \Omega^{0,q}$ satisfies $\overline{\partial}\alpha = 0$ with $q \geqslant 1$. Then there exists a $(0, q-1)$-form β on a neighbourhood of m such that $\overline{\partial}\beta = \alpha$.

(iii) Explain why this $\overline{\partial}$-Poincaré lemma implies that the complex

$$0 \to \Omega^{0,0} \to \Omega^{0,1} \to \Omega^{0,2} \to \cdots \to \Omega^{0,n} \to 0$$

is a *resolution* of $\mathcal{O}$, and that the Dolbeault cohomology groups $E_1^{0,q}$ coincide with the Čech cohomology groups of $\mathcal{O}$.

Example 1.27 A manifold X has a global basis $\{e^1, e^2, e^3, e^4\}$ of 1-forms with the property that

$$de^i = \begin{cases} 0, & i = 1, 2, \\ e^1 \wedge e^2, & i = 3, \\ e^1 \wedge e^3, & i = 4. \end{cases} \tag{1.28}$$

We shall show that there exists no *complex* structure J on X with the property that $Je_i = \sum_j J_i^j e_j$ with J_i^j constant.

Let T^* denote the real 4-dimensional space spanned by the e^i. Under the assumption that there exists such a J, the subspace $\langle e^1, e^2, e^3\rangle_{\mathbb{C}}$ of $T^*_{\mathbb{C}}$ must contain a non-zero $(1,0)$ form α relative to J. If $d\alpha = 0$ then $\alpha \in \langle e^1, e^2\rangle_{\mathbb{C}}$, and

$$e^1 \wedge e^2 \in \Lambda^{1,1}. \tag{1.29}$$

But (1.29) is also valid if $d\alpha \neq 0$, since then $e^1 \wedge e^2$ has no $(0,2)$-component and (being real) no $(2,0)$ component. Now (1.29) implies that $Je^1 \wedge Je^2 = e^1 \wedge e^2$ and that the subspace $\langle e^1, e^2\rangle$ is J-invariant. But the same argument with e^3 replaced by e^4 would give that $\langle e^1, e^3\rangle$ is J-invariant, which is impossible.

Structures on Lie groups

The last exercise is one about a Lie group in disguise. The equations (1.28) determine a Lie algebra structure on any tangent space T to X, and it follows that we may take X to be a corresponding (nilpotent) group. The conclusion is that this Lie group admits no left-invariant complex structure. By contrast, left-invariant complex structures exist on any compact Lie group of even dimension (references are given below). We shall illustrate the case of $SU(2)\times SU(2)$ in this subsection, that was motivated by papers such as [87] describing the deformation of invariant complex structures.

A left-invariant complex structure on a Lie group G is determined by an almost-complex structure J on its Lie algebra $\mathfrak{g}$ satisfying $N=0$ in which the brackets of (1.20) are now interpreted in a purely algebraic sense. The i eigenspace $\mathfrak{g}^{1,0}$ of J is always a *complex Lie algebra*, but the Lie group G itself is not in general complex. The integrability condition is satisfied in the following special cases:

(i) $[JX,Y]=J[X,Y]$. In this case we may write $J=i$ so $(\mathfrak{g},i)$ is a complex Lie algebra isomorphic to $\mathfrak{g}^{1,0}$, and G *is* a complex Lie group.

(ii) $[JX,JY]=[X,Y]$. This condition is equivalent to asserting that d maps the subspace Λ of $(1,0)$-forms $\mathfrak{g}_{\mathbb{C}}$ into $\Lambda^{1,1}$, or that $\mathfrak{g}^{1,0}$ is an *abelian* Lie algebra.

From now on, let $G=SU(2)\times SU(2)$; as a manifold G is the same thing as $S^3\times S^3$. The space of left-invariant 1-forms on the first $SU(2)$ factor is modelled on the Lie algebra $\mathfrak{su}(2)$, and it follows that there is a global basis $\{e^1,e^2,e^3\}$ of 1-forms with the property that

$$de^1=e^{23},\quad de^2=e^{31},\quad de^3=e^{12}.$$

Fix a similar basis $\{e^4,e^5,e^6\}$ of left-invariant 1-forms for the second $SU(2)$ factor. Left-invariant almost-complex structures on G are determined by almost-complex structures on the vector space $\mathbb{R}^6$ spanned by the e^i. As an example, consider the almost-complex structure J_0 whose space Λ of $(1,0)$-forms is spanned by

$$\sigma^1=e^1+ie^2,\quad \sigma^2=e^3+ie^4,\quad \sigma^3=e^5+ie^6.$$

It is easy to check that $d\sigma^i$ has no $(0,2)$-component for each i, so that J_0 is integrable.

There is nothing terribly special about J_0. Fix the inner product for which $\{e^i\}$ is an orthonormal basis, and set $U=\langle e^1,e^2,e^3\rangle$, $V=\langle e^4,e^5,e^6\rangle$. Then J_0 belongs to a family of complex structures parametrized by

$$(\mathrm{u},\mathrm{v})\in S^2\times S^2\subset U\times V, \tag{1.30}$$

which is the orbit of J_0 under right translation by $SU(2)\times SU(2)$. The ordered pair (u,v) determines the structure with $(1,0)$-forms $\{\mathrm{u}',\mathrm{u}+i\mathrm{v},\mathrm{v}'\}$ where

$$\begin{array}{l}\mathrm{u}' \text{ is an isotropic vector in } \langle e^1,e^2,e^3\rangle_{\mathbb{C}} \text{ orthogonal to u, and}\\ \mathrm{v}' \text{ is an isotropic vector in } \langle e^4,e^5,e^6\rangle_{\mathbb{C}} \text{ orthogonal to v.}\end{array} \tag{1.31}$$

In this way, J_0 corresponds to $\mathrm{u}=e^3$ and $\mathrm{v}=e^4$.

Consider now deformations of the complex structure J_0 without reference to an inner product. Suppose that J is an almost-complex structure whose space Λ of $(1,0)$-forms is spanned by $\alpha^1,\alpha^2,\alpha^3$, and set

$$\eta=\alpha^{123}\wedge\overline{\alpha}^{123},\qquad \xi=\alpha^{123}\wedge\overline{\sigma}^{123}$$

($\overline{\alpha}^{123}$ is short for $\overline{\alpha}^1\wedge\overline{\alpha}^2\wedge\overline{\alpha}^3$ etc). The necessary condition $\Lambda\cap\overline{\Lambda}=\{0\}$ is equivalent to $\eta\neq 0$. On the other hand, we may regard the set of J satisfying $\xi\neq 0$ as a large affine neighbourhood of J_0, on which row echelon reduction allows us to take

$$\begin{aligned}\alpha^1&=\sigma^1+r\overline{\sigma}^1+s\overline{\sigma}^2+t\overline{\sigma}^3,\\ \alpha^2&=\sigma^2+u\overline{\sigma}^1+v\overline{\sigma}^2+w\overline{\sigma}^3,\\ \alpha^3&=\sigma^3+x\overline{\sigma}^1+y\overline{\sigma}^2+z\overline{\sigma}^3,\end{aligned}$$

with $r,s,t,u,v,w,x,y,z\in\mathbb{C}$. It is convenient to let X denote the 3×3 matrix formed by these 9 coefficients as displayed. The correspondence $J\leftrightarrow X$ is then one-to-one, and $J_0\leftrightarrow 0$. Structures (such as $-J_0$) for which $\xi=0$ can only be obtained by letting some of the entries of X become infinite.

Proposition 1.32 *Any left-invariant complex structure on G admits a basis $\{\mathrm{u}',\mathrm{u}+\tau\mathrm{v},\mathrm{v}'\}$ of $(1,0)$-forms satisfying (1.30) and (1.31), with $\tau\in\mathbb{C}\setminus\mathbb{R}$.*

Proof By Lemma 1.19, we require that each $d\alpha^i$ have no $(0,2)$-component relative to J. This is equivalent to the equations

$$d\alpha^i\wedge\alpha^{123}=0,\quad i=1,2,3,\tag{1.33}$$

since wedging with the $(3,0)$-form α^{123} annihilates everything but the $(0,2)$-component of $d\alpha^i$. Computations show that this implies that $v\neq 1$. Furthermore, setting $a=v-1$,

(i) $$a^2X=\begin{pmatrix} s^2(a+1) & sa^2 & swa\\ sa(a+2) & a^2(a+1) & wa^2\\ sw(a+2) & wa(a+2) & w^2(a+1)\end{pmatrix},$$

(ii) $$\eta=\frac{-8i(1-|v|^2)(|a|^2+|s|^2)^2(|a|^2+|w|^2)^2}{|a|^4}\,e^{12\cdots 6}.$$

We may now define

$$\begin{aligned}\mathrm{u}'&=a(a\alpha^1-s\alpha^2)=(a^2-s^2)e^1+i(a^2+s^2)e^2-2as\,e^3,\\ \mathrm{v}'&=a(w\alpha^2-a\alpha^3)=2iaw\,e^4-(a^2+w^2)e^5-i(a^2-w^2)e^6,\end{aligned}$$

and τ can be expressed as a function of v. $\square$

The sign of η is determined by $|v|$, and the set $\mathcal{C}$ of invariant complex structures inducing the same orientation as J_0 has $|v| < 1$. The above proposition allows us to identify $\mathcal{C}$ with $\Delta \times (S^2 \times S^2)$, where Δ is the unit disc in $\mathbb{C}$. If $v = 0$ then X is a skew-symmetric matrix and we recover the subset $S^2 \times S^2$ described above.

The fact that any even-dimensional compact Lie group admits a complex structure is due to Samelson [97] and Wang [104]. This theory was generalized by Snow [99], who described the moduli space of so-called 'regular' invariant complex structures on any reductive Lie group such as $GL(2m, \mathbb{R})$. The nilpotent case is covered by [32] and the author's paper [95].

Exercises 1.34 (i) The manifold $S^3 \times S^1$ can be identified with the Lie group $SU(2) \times U(1)$. As such, it has a global basis $\{e^1, e^2, e^3, e^4\}$ of left-invariant 1-forms such that

$$de^1 = e^{23}, \quad de^2 = e^{31}, \quad de^3 = e^{12}, \quad de^4 = 0.$$

Deduce that $S^3 \times S^1$ has a left-invariant hypercomplex structure, but no abelian complex structures.

(ii) Let $\mathfrak{g}$ be the Lie algebra of an even-dimensional compact Lie group, and let $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus [\![\mathfrak{g}_\alpha]\!]$ be a root space decomposition in which the big sum is taken over a choice of positive roots. Noting that each real 2-dimensional space $[\![\mathfrak{g}_\alpha]\!]$ carries a natural almost-complex structure, show that any almost-complex structure on $\mathfrak{t}$ can be extended to an almost-complex structure J on $\mathfrak{g}$ with $N = 0$.

(iii) Compute the spaces and maps in the array (1.24) for the complex manifold $(S^3 \times S^3, J_0)$, and determine $\sum\limits_{p=0}^{3} h^{p,3-p}$.

2 Almost-Hermitian metrics

Let M be a smooth manifold. A Riemannian metric on M is the assignment of a smoothly-varying inner product (that is, a positive-definite symmetric bilinear form) on each tangent space. We write $g(X, Y)$ (rather than $\langle X, Y\rangle$) to denote the inner product of two tangent vectors or fields X, Y. Thus, g is a smooth section over M of $T^* \otimes T^*$, where T^* is the cotangent bundle, and indeed $g \in \Gamma(\bigodot^2 T^*)$, where $\bigodot^2 T^*$ is the symmetrized tensor product.

Suppose that M is a Riemannian manifold of dimension $2n$. An almost-complex structure J on M is said to be *orthogonal* if $g(JX, JY) = g(X, Y)$. Note that this condition depends only on the conformal class of g. Indeed, when $n = 1$ such a J is identical to the oriented conformal structure determined by g; given this, J is essentially 'rotation by 90^o'. An almost-complex structure J is orthogonal if and only if its space Λ of $(1, 0)$ forms is totally isotropic at each point. The triple (M, g, J) with J orthogonal is called an *almost-Hermitian manifold*.

We shall first define the fundamental 2-form of an almost-Hermitian manifold, and then study the special case in which the manifold is Kähler. After a summary of some relevant Hodge theory, we consider the parametrization of compatible complex structures on a given even-dimensional Riemannian manifold.

The Kähler condition

Given an almost-Hermitian manifold (M,g,J), the tensor

$$\omega(X,Y)=g(JX,Y) \tag{2.1}$$

is a 2-form on M. This 2-form is non-degenerate in the sense that it defines an isomorphism $T\to T^*$ at each point. Equivalently, the volume form

$$\upsilon=\frac{\omega^n}{n!} \tag{2.2}$$

of M is nowhere zero. To check the constants, suppose that $n=3$ and that $g=\sum_{i=1}^3 e^i\otimes e^i$ at a fixed point $m\in M$. A standard almost-complex structure is determined by setting $\omega=e^{12}+e^{34}+e^{56}$, with the convention $e^{ij}=e^i\wedge e^j=e^i\otimes e^j-e^j\otimes e^i$. Then $\omega^3=6e^{123456}=6\upsilon$.

A standard almost-complex structure J on $\mathbb{R}^{2n}$ is the *linear transformation* described by the matrix

$$\mathbb{J}=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix} \tag{2.3}$$

where I is the $n\times n$ identity matrix. The stabilizer of J in $GL(2n,\mathbb{R})$ is isomorphic to $GL(n,\mathbb{C})$; it is the set of matrices A for which $A^{-1}\mathbb{J}A=\mathbb{J}$ or $A\mathbb{J}=\mathbb{J}A$. The matrix $\mathbb{J}$ may also be regarded as that of a standard non-degenerate skew bilinear form ω_0 on $\mathbb{R}^{2n}$. It follows that the choice of ω_0 determines a reduction to the subgroup $Sp(2n,\mathbb{R})$, determined by the set of matrices such that $A^T\mathbb{J}A=\mathbb{J}$.

The metric g determines a reduction to the orthogonal group $O(2n)$ and the intersection of any two of the subgroups $GL(n,\mathbb{C})$, $Sp(2n,\mathbb{R})$, $O(2n)$ is isomorphic to the unitary group $U(n)$. An almost-Hermitian structure on a manifold is therefore determined by a reduction of its principal bundle of frames to $U(n)$.

The 2-form ω defines an 'almost-symplectic' structure on M, which is called 'symplectic' if $d\omega=0$. The latter condition is analogous to the integrability of J, since the Darboux theorem asserts that $d\omega=0$ if and only if there exist real coordinates $x^1,\dots,x^n,y^1,\dots,y^n$ such that

$$\omega=\sum_{r=1}^n dx^r\wedge dy^r \tag{2.4}$$

(compare (1.11)) in a neighbourhood U of any given point.

Definition 2.5 *An almost-Hermitian manifold M is said to be Hermitian if J is integrable, and Kähler if in addition $d\omega = 0$.*

Since

$$\omega(X - iJX, Y - iJY) = i\,g(X - iJX, Y - iJY) = 0, \quad X, Y \in T,$$

we can assert that ω is a $(1,1)$-form. Relative to local holomorphic coordinates on a Hermitian manifold, we may write

$$\omega = \tfrac{1}{2} i \sum_{k,l} g_{kl} dz^k \wedge d\overline{z}^l. \tag{2.6}$$

Applying (2.1) backwards, we see that this gives rise to a corresponding expression

$$g = \tfrac{1}{2} \sum_{k,l} \omega_{kl} dz^k \odot d\overline{z}^l$$

of the Riemannian metric, with the convention

$$dz \odot d\overline{z} = dz \otimes d\overline{z} + d\overline{z} \otimes dz = 2(dx \otimes dx + dy \otimes dy).$$

Lemma 2.7 *Let M be a Kähler manifold and $m \in M$. Then there exists a real-valued function ϕ on a neighbourhood of m such that $\omega = \frac{1}{2} i\partial\overline{\partial}\phi$, so that $\omega_{kl} = \partial^2\phi/\partial z^k \partial\overline{z}^l$.*

Proof Since $d\omega = 0$, there exists a $(1,0)$-form α on a neighbourhood of m such that $\omega = d(\alpha + \overline{\alpha})$. But then $\overline{\partial}\overline{\alpha} = 0$ and, by Exercise 1.26(ii), there exists a complex-valued function f on a possibly smaller neighbourhood such that $\overline{\partial} f = \overline{\alpha}$. Setting $\phi = \frac{1}{2} i(\overline{f} - f)$ gives

$$2\omega = d(\partial\overline{f} + \overline{\partial} f) = \overline{\partial}\partial\overline{f} + \partial\overline{\partial} f = i\partial\overline{\partial}\phi,$$

as required. □

Example 2.8 The most obvious example of such a 'Kähler potential' is $\phi = \sum_{r=1}^{n} |Z^r|^2$ on $\mathbb{R}^{2n} = \mathbb{C}^n$. This gives rise to the flat Kähler metric

$$\omega = \tfrac{1}{2} i \sum_{r=1}^{n} dZ^r \wedge d\overline{Z}^r.$$

The standard Kähler metric on $\mathbb{CP}^n$ is constructed most invariantly by starting from the function $\psi = \log \sum_{r=0}^{n} |Z^r|^2$ on $\mathbb{C}^{n+1} \setminus \{0\}$. Then

$$\partial\overline{\partial}\psi = \partial\overline{\partial} \log\left(1 + \sum_{r=1}^{n} |z^r|^2\right) = \pi^*\omega,$$

where ω has the form (2.1) in local coordinates on U_0. However, the construction shows that ω is well defined globally on $\mathbb{CP}^n$. One recovers, for $n=1$, the form and metric

$$\omega = \frac{i\,dz \wedge d\overline{z}}{2(1+|z|^2)^2}, \qquad g = \frac{dx^2+dy^2}{(1+x^2+y^2)^2}$$

with constant positive Gaussian curvature.

On a compact Kähler manifold M of real dimension $2n$, the powers ω^k of ω represent non-zero cohomology classes in $H^{2k}(M,\mathbb{R})$ for all $k \leqslant n$. This is because a global equation $\omega^n = d\sigma$ would imply that (by Stokes' theorem) that $\int_M \omega^n = 0$, contradicting (2.2). Since $H^2(\mathbb{CP}^n,\mathbb{R})$ is well known to be 1-dimensional, the 2-form ω constructed above can be normalized so as to belong to $H^2(\mathbb{CP}^n,\mathbb{Z})$. The power of the Kähler condition is that any complex submanifold M of $\mathbb{CP}^n$ is automatically Kähler, since ω on $\mathbb{CP}^n$ pulls back to a closed form on M compabible with the induced metric and complex structure. On the other hand, it is known that a necessary and sufficient condition for a compact complex manifold M to satisfy Definition 1.6 is that it possess a Kähler metric ω with $[\omega] \in H^2(M,\mathbb{Z})$ (see for example [54]).

The restriction of the standard dot product on $\mathbb{R}^n$ to any subspace is an inner product, so any submanifold of $\mathbb{R}^n$ has an induced Riemannian metric. A wider class of metrics can be constructed by considering certain types of submanifolds of a real vector space endowed with a bilinear form of mixed signature. A classical example of this is the pseudosphere $-x^2-y^2+z^2=1$ of negative Gaussian curvature isometrically embedded in the 'Lorentzian' space $\mathbb{R}^{1,2}$. Additional structures on the ambient vector space can $\mathbb{R}^n$ can sometimes be used to induce an almost-Hermitian structure on a hypersurface or submanifold.

Example 2.9 Identify $\mathbb{R}^7$ with the space of imaginary Cayley numbers, which is endowed with a (non-associative) product $\times$. Then any hypersurface M of $\mathbb{R}^6$ has an almost-complex structure J, compatible with the induced metric, defined by $JX = \mathbf{n} \times X$ where $X \in T_mM$ and $\mathbf{n}$ is a consistently-defined unit normal vector at m. The Nijenhuis tensor of J can be related to the second fundamental form of M, and Calabi proved that J is integrable if and only if M is minimal [28; 91]. A generalization of this phenomenon for submanifolds of $\mathbb{R}^8$ is described in [21].

There is also a Kähler version of Definition 1.12:

Definition 2.10 *A manifold is hyperkähler if it admits a Riemannian metric g which is Kähler relative to complex structures I_1, I_2, I_3 satisfying (1.13).*

The significance of the existence of such complex structures will (at least in 4 dimensions) will become clearer after Proposition 2.25. The integrability requires that the 2-form ω_i associated to I_i is closed for each i. If we fix I_1 then $\eta = \omega_2 + i\omega_3$ is a $(2,0)$-form, which is closed and so holomorphic since the $(2,1)$-form $\overline{\partial}\eta$ vanishes. In fact, a hyperkähler manifold M has dimension $4n$, and (choosing an appropriate basis $\{\alpha^1,\dots,\alpha^{2n}\}$ of $(1,0)$-forms at each point) $\eta^n=\alpha^{12\cdots n}$ trivializes $\Lambda^{2n,0}$, so M is holomorphic symplectic. Conversely, with the same algebraic set-up, the closure of η implies that I_1 is integrable; this is because

$$0=d(\eta^n)^{n-1,2}=(d\alpha^1)^{0,2}\wedge\alpha^{23\cdots n}+(d\alpha^2)^{0,2}\wedge\alpha^{3\cdots n1}+\cdots$$

and $(d\alpha^i)^{0,2}$ for all i.

For future reference, we describe the standard 2-forms associated to a flat hyperkähler structure on $\mathbb{R}^{4n}$. Consider coordinates (x^r,y^r,u^r,v^r) with $1\leqslant i\leqslant n$. Then

$$\begin{cases}\omega_1=\displaystyle\sum_{r=1}^n(dx^r\wedge dy^r+du^r\wedge dv^r),\\ \omega_2=\displaystyle\sum_{r=1}^n(dx^r\wedge du^r+dv^r\wedge dy^r),\\ \omega_3=\displaystyle\sum_{r=1}^n(dx^r\wedge dv^r+dy^r\wedge du^r).\end{cases}\tag{2.11}$$

Exercises 2.12 (i) Show that the mapping $A\longmapsto A\mathbb{J}$ identifies the Lie algebra $\mathfrak{sp}(2n,\mathbb{R})$ with the space of symmetric matrices. Show that $\mathfrak{sp}(2,\mathbb{R})$ is isomorphic to the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of real 2×2 matrices of trace zero.

(ii) Further to Example 2.8, show that the function $\phi+\log\phi$, where $\phi=\sum_{r=0}^n|Z^r|^2$, is the potential for a Kähler metric that extends to the blow-up of $\mathbb{C}^n$ at the origin.

(iii) Let $n=2$. The subgroup of $GL(8,\mathbb{R})$ preserving all three 2-forms (2.11) is exactly $Sp(2)$. Determine the stabilizer of each of the two 2-forms

$$\Phi_\pm=\omega_1\wedge\omega_1+\omega_2\wedge\omega_2\pm\omega_2\wedge\omega_3,$$

referring to [24; 92] if necessary.

Summary of Hodge theory

It turns out that a global version of Lemma 2.7 is valid on a compact Kähler manifold. In order to explain this, we sketch the essentials of Hodge theory in the context of Dolbeault cohomology. More details can be found in many standard texts [54; 57; 105; 106; 107].

Let M be a complex manifold of real dimension $2n$. An inner product on the complex space $\Omega^{p,q}$ is defined by

$$\langle\!\langle\alpha,\beta\rangle\!\rangle=\int_M g(\alpha,\overline{\beta})\upsilon=\int\alpha\wedge*\overline{\beta}=\int\overline{\beta}\wedge*\alpha,$$

where g is the Riemannian metric (extended as a complex *bilinear* form), and

$$*\colon\Lambda^{p,q}\to\Lambda^{n-q,n-p}$$

is the complexification of a *real* isometry satisfying $*^2\alpha = (-1)^{p+q}\alpha$. The double brackets emphasize the global nature of the pairing. If $\alpha \in \Omega^{p,q}$ and $\beta \in \Omega^{p,q-1}$ then

$$d(\beta \wedge *\overline{\alpha}) = \overline{\partial}(\beta \wedge *\overline{\alpha}) = \overline{\partial}\beta \wedge *\overline{\alpha} - (-1)^{p+q}\beta \wedge \overline{\partial}(*\overline{\alpha}),$$

and it follows that

$$\langle\!\langle \overline{\partial}\beta, \alpha \rangle\!\rangle = -\langle\!\langle \beta, *\partial * \alpha \rangle\!\rangle.$$

Although the space $\Omega^{p,q}$ of smooth forms is not complete for the norm defined above, we may regard $\overline{\partial}^* = - * \partial *$ as the adjoint of the mapping (2.13) below.

Fix a Dolbeault cohomology class $c \in H^{p,q}_{\overline{\partial}}$. To represent c uniquely, one seeks a $\overline{\partial}$-closed (p,q)-form α of least norm, thus *orthogonal* to the image of

$$\overline{\partial}\colon \Omega^{p,q-1} \to \Omega^{p,q}. \tag{2.13}$$

Since $\langle\!\langle \alpha, \overline{\partial}\gamma \rangle\!\rangle = \langle\!\langle \overline{\partial}^*\alpha, \gamma \rangle\!\rangle$ for all γ, we therefore need $\overline{\partial}^*\alpha = 0$. Now $\langle\!\langle \alpha, \Delta_{\overline{\partial}}\alpha \rangle\!\rangle = \|\overline{\partial}\alpha\|^2 + \|\overline{\partial}^*\alpha\|^2$, where

$$\Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} \tag{2.14}$$

is the $\overline{\partial}$-*Laplacian*, so we require α to belong to the space

$$\mathcal{H}^{p,q} = \{\alpha \in \Omega^{p,q} : \Delta_{\overline{\partial}}\alpha = 0\}$$

of *harmonic* (p,q)-forms. It also follows from above that α is orthogonal to the image $\Delta_{\overline{\partial}}(\Omega^{p,q})$ if and only if α is harmonic.

Suppose for a moment that $\Omega^{p,q}$ were a finite-dimensional vector space. This is not as unrealistic as it seems, because examples we have considered earlier based on Lie groups and or their quotients possess subcomplexes consisting of *invariant* differential forms with constant coefficients, so one could restrict to these to obtain a more restricted type of cohomology. In this situation, $\Omega^{p,q}$ is the direct sum of $\Delta_{\overline{\partial}}\Omega^{p,q}$ and its orthogonal complement $\mathcal{H}^{p,q}$. The Hodge theorem asserts that this applies in the general case:

Theorem 2.15 *Let M be a compact complex manifold. Then $\mathcal{H}^{p,q}$ is finite-dimensional, and there is a direct sum decomposition $\Omega^{p,q} = \mathcal{H}^{p,q} \oplus \Delta_{\overline{\partial}}\Omega^{p,q}$.*

It follows that, modulo a finite-dimensional space, the Laplacian is invertible. For given $\alpha \in \Omega^{p,q}$, there exists $G\alpha \in \Omega^{p,q}$ such that $(\Delta_{\overline{\partial}}G\alpha - \mathbb{1})\alpha$ is harmonic. The mapping G is called the *Green's operator*. One has additional orthogonal direct sums

$$\begin{aligned} \Omega^{p,q} &= \mathcal{H}^{p,q} \oplus \overline{\partial}(\overline{\partial}^*\Omega^{p,q}) \oplus \overline{\partial}^*(\overline{\partial}\Omega^{p,q}) \\ &= \mathcal{H}^{p,q} \oplus \overline{\partial}\Omega^{p,q-1} \oplus \overline{\partial}^*\Omega^{p,q+1}. \end{aligned}$$

To see the second equality, observe that if for example $\beta \in \Omega^{p,q-1}$ then $\beta - \Delta_{\overline{\partial}}G\beta$ is harmonic, and so $\overline{\partial}\beta = \overline{\partial}\overline{\partial}^*\gamma$ where $\gamma = \overline{\partial}G\beta$.

Corollary 2.16 $H^{p,q}_{\overline{\partial}} \cong \mathcal{H}^{p,q}$.

Proof If $\gamma \in \Omega^{p,q}$ is $\overline{\partial}$-closed then γ is orthogonal to $\overline{\partial}^*\Omega^{p,q+1}$. Thus, the kernel of $\overline{\partial}$ on $\Omega^{p,q}$ equals $\mathcal{H}^{p,q} \oplus \overline{\partial}\Omega^{p,q-1}$. It follows that each cohomology class has a unique harmonic representative. $\square$

Corollary 2.17 $H^{p,q}_{\overline{\partial}} \cong H^{n-p,n-q}_{\overline{\partial}}$.

Proof Since $\Delta_{\overline{\partial}} * \overline{\alpha} = \overline{*\Delta_{\overline{\partial}}\alpha}$, the composition of $*$ with conjugation determines an isomorphism from $\mathcal{H}^{p,q}$ to $\mathcal{H}^{n-p,n-q}$. $\square$

We shall now return to the Kähler condition. From (2.2) we may deduce that $*\omega = \omega^{n-1}/(n-1)!$ Moreover, if e^1 is a unit 1-form with $Je^1 = -e^2$, then $e^1 \wedge e^2 \wedge \omega^{n-1} = (n-1)!\,\upsilon$, so $*e^1 = e^2 \wedge \omega^{n-1}/(n-1)!$ It follows that, if f is a function,

$$\begin{aligned}\overline{\partial}^*(f\omega) = -*\partial*(f\omega) &= -\tfrac{1}{(n-1)!} * \partial(f\omega^{n-1}) \\ &= -\tfrac{1}{(n-1)!} * (\partial f \wedge \omega^{n-1}) \\ &= J(\partial f) = i\partial f.\end{aligned}$$

This is a special case of

Lemma 2.18 *Let* $\alpha \in \Omega^{p,q}$. *Then* $\overline{\partial}^*(\omega \wedge \alpha) - \omega \wedge \overline{\partial}^*\alpha = i\partial\alpha$.

We omit the general proof, which is most easily carried out with connections and theory developed in Section 3. The lemma is illustrated by the diagram

$$\begin{array}{ccccc} & & & & \Omega^{p+1,q+1} \\ & & \nearrow & \downarrow & \\ & \Omega^{p,q} & \longrightarrow & \Omega^{p+1,q} & \\ & \downarrow & \nearrow & & \\ \Omega^{p,q-1} & & & & \end{array}$$

Let $L\colon \bigwedge^k T^* \to \bigwedge^{k+2} T^*$ denote the operation of wedging with ω, and

$$\Lambda = L^* = (-1)^k * L *$$

its adjoint relative to $\langle\!\langle .,. \rangle\!\rangle$ (equivalently g). Then

$$\begin{aligned} [\overline{\partial}^*, L] &= i\partial, \qquad & [\partial^*, L] &= -i\overline{\partial}, \\ [\overline{\partial}, \Lambda] &= i\partial^*, \qquad & [\partial, \Lambda] &= -i\overline{\partial}^*; \end{aligned}$$

the first equation is a restatement of the lemma, and the others immediate consequences. Substituting the last two lines into an expansion of

$$\Delta_d = dd^* + d^*d = (\partial + \overline{\partial})(\partial + \overline{\partial})^* + (\partial + \overline{\partial})^*(\partial + \overline{\partial})$$

to eliminate the last two terms on the right-hand side reveals that

$$\Delta_d = \partial\partial^* + \partial^*\partial + \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} = \Delta_\partial + \Delta_{\overline{\partial}},$$

and that $\Delta_\partial = \Delta_{\overline{\partial}}$. Hodge theory for the exterior derivative now yields

Theorem 2.19 *On a compact Kähler manifold $H^k(M,\mathbb{R})$ is isomorphic to*

$$\{\alpha\in\Gamma(\textstyle\bigwedge^k T^*M):\Delta_d\alpha=0\}=\bigoplus_{p+q=k}\mathcal{H}^{p,q},$$

and there is equality in (1.25).

Complex conjugation gives an isomorphism $\mathcal{H}^{p,q}\cong\overline{\mathcal{H}^{q,p}}$, and so $h^{p,q}=h^{q,p}$. In particular, the Betti numbers b_{2i+1} are all even. Moreover the mapping

$$L^{n-k}: H^k(M,\mathbb{R})\to H^{2n-k}(M,\mathbb{R}) \tag{2.20}$$

is an isomorphism; this is the so-called *hard Lefschetz property*. The failure of (2.20) underlies almost all known examples of manifolds that do not admit a Kähler metric.

The goal of this subsection is achieved by

Lemma 2.21 *Let M be a compact Kähler manifold, and let $\alpha\in\Omega^{p,q}$ be cloised $(p,q\geqslant 1)$. Then $\alpha=\partial\overline{\partial}\beta$ for some $\beta\in\Omega^{p-1,q-1}$.*

Proof In the notation (2.14), we know that $\alpha-\Delta_{\overline{\partial}}G\alpha$ is annihilated by $\Delta_{\overline{\partial}}=\frac{1}{2}\Delta_d$. Thus

$$0=d\alpha-d\Delta_{\overline{\partial}}G\alpha=-(\partial+\overline{\partial})\overline{\partial}\,\overline{\partial}^*G\alpha=-\partial\overline{\partial}(\overline{\partial}^*G\alpha),$$

using the fact that $\overline{\partial}G=G\overline{\partial}$. □

Known obstructions to the existence of Kähler metrics on compact manifolds stem from (2.20) or this $\partial\overline{\partial}$ lemma. Either can be used to establish the *formality* of the deRham cohomology of Kähler manifolds [33]. More recent references to this topic can be found in [78; 101].

Exercises 2.22 (i) Let M be a compact complex manifold of dimension n, and let α be a k-form satisfying $\overline{\partial}\alpha=0$. Prove that if M is Kähler *or* if $k=n-1$ then one may conclude that $d\alpha=0$.

(ii) Describe the set $\mathcal{S}$ of all symplectic forms on the manifold in Example 1.22 of the type $\omega=\sum\omega_{ij}e^i\wedge e^j$ with ω_{ij} constant. Is $\mathcal{S}$ connected? What is its dimension?

(iii) Compute the action of d on the basis $\{e^i\wedge e^j: i<j\}$ of 2-forms of X in Example 1.27. What can you say about the *Betti numbers* of Y? Is the mapping $H^1(Y,\mathbb{R})\to H^3(Y,\mathbb{R})$ induced by wedging with the symplectic form $\omega=e^{14}+e^{23}$ an isomorphism?

Orthogonal complex structures

If we are given a complex manifold (M,J), there is no difficulty in choosing a Hermitian metric. Namely, pick any metric h and then define

$$g(X,Y)=h(X,Y)+h(JX,JY); \tag{2.23}$$

this renders J orthogonal. One can do the same sort of thing if M is hypercomplex by defining

$$g(X,Y) = \sum_{k=0}^{3} h(I_k X, I_k Y),$$

where $I_0 = \mathbb{1}$. For then g will be 'hyperhermitian' in the sense that $g(JX,JY) = g(X,Y)$ whenever $J = \sum a_r I_r$ with $\sum {a_r}^2 = 1$.

The reverse problem has a very different character. Given an oriented Riemannian manifold (M,g) of even dimension $2n$, it is not in general a straightforward job to find a complex structure J for which (M,g,J) is Hermitian. In the presence of an assigned metric, any such J is referred to as an *orthogonal complex structure*, or OCS for short.

Let Z_n denote the set of orthogonal almost-complex structures on $\mathbb{R}^{2n}$ compatible with a standard orientation. The group $SO(2n)$ acts transitively on Z_n, and using the notation (2.3),

$$Z_n = \{A^{-1}\mathbb{J}A : A \in SO(2n)\}.$$

If $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ where each A_i is $n \times n$ then $A\mathbb{J} = \mathbb{J}A$ iff $A_1 = A_4$ and $A_2 = -A_3$, and the orthogonality implies that $A_1 + iA_2$ is a unitary matrix. Thus the stabilizer of $\mathbb{J}$ is isomorphic to $U(n)$, and Z_n can be identified with the homogeneous space $SO(2n)/U(n)$.

Any almost-complex structure on $\mathbb{R}^{2n}$ is completely determined by the corresponding space $\Lambda = \Lambda^{1,0}$ of $(1,0)$-forms in (1.4). Given Λ, we may obviously define J by $J(v + \overline{w}) = iv - i\overline{w} = iv + \overline{iw}$. The mapping

$$J \mapsto \Lambda$$

identifies Z_n with one component of the subset of maximal isotropic subspaces of $\mathbb{C}^{2n}$ in the Grassmannian $\mathbb{G}\mathrm{r}_n(\mathbb{C}^{2n})$. This gives Z_n a natural complex structure. In fact, $SO(2n)/U(n)$ is a Hermitian symmetric space and admits a Kähler metric compatible with this complex structure.

Let $\{e_1,\dots,e_{2n+2}\}$ be an oriented orthonormal basis of $\mathbb{R}^{2n+2}$. Define a mapping

$$\begin{array}{l} Z_{n+1} \\ \big\downarrow \pi \\ S^{2n} \subset \langle e_2, \dots e_{2n+2}\rangle, \end{array} \tag{2.24}$$

where S^{2n} is the sphere, by setting $\pi(J) = Je_1$. Then $\pi^{-1}(e_2)$ is the set of oriented orthogonal complex structures on $\langle e_1, e_2\rangle^\perp = T_{\pi(J)}S^{2n}$. Thus (2.24) is a bundle with fibre Z_n. Of course, Z_1 is a point. The next result shows that Z_2 is a 2-sphere.

Proposition 2.25 *π is an isomorphism for $n = 1$.*

Proof Let J be an almost-complex structure compatible with the metric and orientation of $\mathbb{R}^4$. From above we may write

$$-Je_1 = ae_2 + be_3 + ce_4, \quad a^2 + b^2 + c^2 = 1.$$

Passing to the dual basis, the 2-form associated to J is

$$\omega = e^1 \wedge (-Je^1) + f \wedge (-Jf),$$

where f is any unit 1-form orthogonal to both e^1 and Je^1. Since $be^2 - ae^3$ and $ce^2 - ae^4$ are both orthogonal to Je^1,

$$f \wedge (-Jf) = \frac{1}{a}(be^2 - ae^3) \wedge (ce^2 - ae^4) = ae^{34} + be^{42} + ce^{23}$$

is completely determined. $\square$

This proof shows that the 2-form of any compatible almost-complex structure on $\mathbb{R}^4$ can be written $a\omega_1 + b\omega_2 + c\omega_3$, where $a^2 + b^2 + c^2 = 1$, and

$$\left\{ \begin{aligned} \omega_1 &= e^{12} + e^{34}, \\ \omega_2 &= e^{13} + e^{42}, \\ \omega_3 &= e^{14} + e^{23}. \end{aligned} \right. \tag{2.26}$$

Definition 2.27 *The span of $\omega_1, \omega_2, \omega_3$ is the space Λ^+ of self-dual 2-forms on $\mathbb{R}^4$.*

The above argument shows that Λ^+ depends only on the choice of metric and orientation. A fixed almost-complex structure J gives a splitting

$$\Lambda^2 T^* = [\![\Lambda^{2,0}]\!] \oplus [\Lambda^{1,1}], \tag{2.28}$$

corresponding to a reduction to $U(1) \times SU(2)$. Moreover, $[\![\Lambda^{2,0}]\!]$ is the real tangent space to $S^2 \subset \Lambda^+$ at the point ω associated to J.

If (M, g, J) is an almost-Hermitian 4-manifold then

$$d\omega = \theta \wedge \omega$$

for some 1-form θ (to see this, write $\theta = \sum a_i e^i$). The object θ is (modulo a universal constant) the so-called *Lee form* [76]. Suppose that M is a 4-manifold with a global basis $\{e^1, e^2, e^3, e^4\}$ of 1-forms, so that ω_i is defined by (2.26). Let I_i denote the corresponding almost-complex structure, and θ_i the Lee form, for $i = 1, 2, 3$.

Lemma 2.29 *I_1 is integrable if and only if $\theta_2 = \theta_3$.*

Proof The space of $(1,0)$-forms relative to I_1 is spanned by

$$\alpha^1 = e^1 + ie^2, \quad \alpha^2 = e^3 + ie^4.$$

As in (1.33), I_1 is integrable if and only if

$$\begin{aligned} 0 \;=\; d\alpha^i \wedge (\omega_2 + i\omega_3) &= -\alpha^i \wedge (d\omega_2 + id\omega_3) \\ &= -\alpha^i \wedge (\omega_2 \wedge \theta_2 + i\omega_3 \wedge \theta_3) \\ &= -\tfrac{1}{2}\alpha^i \wedge (\omega_2 - i\omega_3) \wedge (\theta_2 - \theta_3), \end{aligned}$$

since $\alpha^i \wedge (\omega_2 + i\omega_3) = 0$. The result follows. □

This innocent looking lemma has the following important consequences:

(i) If I_1 and I_2 are anti-commuting complex structures then $I_3 = I_1 I_2$ is integrable and M is hypercomplex. In this case, the θ_i coincide and represent a common Lee form.

(ii) If $\theta_2 = 0 = \theta_3$ then $(M, I_1, \omega_2 + i\omega_3)$ is holomorphic symplectic. Moreover, M is hyperkähler if and only if $\theta_i = 0$ for all i.

Similar statements hold in higher dimensions. Refer to [40] for examples of hypercomplex structures in eight real dimensions for which the above theory can easily be applied.

Example 2.30 The following theory was investigated by Barberis [12; 13]. Define real 1-forms by

$$E^1 = e^{-t}dx, \quad E^2 = e^{-t}dv, \quad E^3 = e^{-2t}(du + cx\,dv), \quad E^4 = dt,$$

where $c \in \mathbb{R}$. Then

$$\begin{cases} dE^1 = E^1 \wedge E^4, \\ dE^2 = E^2 \wedge E^4, \\ dE^3 = cE^1 \wedge E^2 + 2E^3 \wedge E^4, \\ dE^4 = 0. \end{cases}$$

Consider the metric

$$g_c = \sum_{i=1}^{4} E^i \otimes E^i = e^{-2t}(dx^2 + dv^2) + e^{-4t}(du + cx\,dv)^2 + dt^2. \tag{2.31}$$

The Lee forms defined by the basis $\{E^i\}$ are

$$\theta_1 = (c-2)e^4, \qquad \theta_2 = \theta_3 = -3e^4,$$

so I_1 is integrable. There are two special cases:

(i) $c = 2$. Then $d\omega_1 = 0$ and (g_2, I_1, ω_1) is Kähler.

(ii) $c=-1$. Then I_2 and I_3 are also integrable, and g_{-1} is hypercomplex.

It is known that g_2 is isometric to the symmetric metric on complex hyperbolic space $\mathbb{C}H^2$. Indeed, x,t,u,v are coordinates on a solvable Lie group that acts simply transitively on $\mathbb{C}H^2$, and the E^i generate the dual of its Lie algebra $\mathfrak{g}$. Being Einstein, g_2 is one of the metrics that crops up in Jensen's classification [61]. Here though, we shall focus on g_{-1}, and show that it is *conformally* hyperkähler.

Setting $c=-1$ and $s=e^t$ in (2.31) gives

$$s^3 g_{-1} = s(dx^2+dv^2+ds^2)+\frac{1}{s}(du-x\,dv)^2.$$

This is a very special case of the Gibbons-Hawking ansatz, that classifies all hyperkähler metrics with a triholomorphic S^1 action [48]. It is easy to see directly that s^3g_{-1} is hyperkähler. Guided by (2.11), define

$$\left\{\begin{aligned} \omega_1 &= s\,dx\wedge dv+(du-x\,dv)\wedge ds,\\ \omega_2 &= dx\wedge(du-x\,dv)+s\,ds\wedge dv,\\ \omega_3 &= s\,dx\wedge ds+dv\wedge(du-x\,dv).\end{aligned}\right. \tag{2.32}$$

Relative to s^3g_{-1}, these are associated to a triple of almost-Hermitian structures I_1,I_2,I_3 for which clearly $d\omega_i=0$ for all i.

In the above description of s^3g_{-1}, one regards the 4-dimensional space as an S^1-bundle over the half space $\mathbb{R}^2\times\mathbb{R}^+$. Then u is a fibre coordinate and $du-x\,dv$ a connection 1-form. The projection to (x,v,s)-space is the so-called *hyperkähler moment mapping* defined by the S^1 group of isometries. The metrics g_2,g_{-1} can also be characterized (amongst all invariant ones on 4-dimensional Lie groups) by the conditions that $W_+=0$ and $W_-\neq 0$ [96]. (See Section 4 for a description of the Weyl tensor W_++W_-.)

Passing to higher dimensions, it can be shown that the space Z_3 of almost-complex structures on $\mathbb{R}^6$ is isomorphic to the projective space $\mathbb{CP}^3$. To visualize this equivalence, one needs a scheme whereby the 4 coordinates of $\mathbb{R}^4$ are linked combinatorially to the 6 coordinates of $\mathbb{R}^6$. Such a 'tetrahedral' isomorphism is described in [2], and exploited in [11], to describe structures on 6-manifolds. Using homogeneous coordinates on $\mathbb{CP}^3$, the four points $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$ can be identified with those almost-complex structures that have 2-forms

$$\pm e^{12}\pm e^{34}\pm e^{56}$$

(with an even number of minus signs to fix the orientation) relative to an orthonormal basis of $\mathbb{R}^6$.

In general, one can show that the image of a section $s\colon S^{2n}\to Z_{n+1}$ is a complex submanifold if and only if the almost-complex structure determined by s is integrable. Since Z_{n+1} is Kähler, the submanifold would itself (and so S^{2n})

have to be Kähler. Since $b_2(S^{2n}) = 0$ for $n > 1$, the sphere S^{2n} cannot admit an OCS unless $n = 1$. Actually, it is well known that S^{2n} does not admit any *almost*-complex structure unless $n = 1$ or 3, so the force of this statement is restricted to S^6. Properties of the Lie group $SO(8)$ enable the total space Z_4 over S^6 to be identified with $SO(8)/(SO(2) \times SO(6))$, itself a complex quadric in $\mathbb{CP}^7$. Note that, as soon as a point is removed, $S^{2n} \setminus \{x\}$ (being conformally equivalent to $\mathbb{R}^{2n}$) does admit OCS's for any n.

The fibration (2.24) can be generalized by replacing the base by any even-dimensional Riemannian manifold M, and taking the associated bundle whose fibre $\pi^{-1}(m)$ (again Z_n) parametrizes compatible almost-complex structures on $T_m M$ [17; 85]. The total space ZM admits two natural almost-complex structures, denoted J_1 and J_2 in [91], which are best defined with the aid of the Levi-Civita connection on M (see Corollary 3.17). Whilst an example of J_1 is provided by the complex structure on $Z_{n+1} = ZS^{2n}$ described above, J_2 is never integrable. On the other hand, if $\psi\colon \Sigma \to ZM$ is a J_2-holomorphic mapping from a Riemann surface Σ, its projection $\pi \circ \psi : \Sigma \to M$ is a harmonic mapping. This universal property characterizes many other types of 'twistor bundles' that have also proved useful in classifying OCS's on symmetric spaces [26; 25].

When M is a 4-manifold, Proposition 2.25 tells us that its twistor space can be identified with the 2-sphere bundle $S(\Lambda^+ T^* M)$. In this case, (2.24) is the celebrated Penrose fibration, and Atiyah-Hitchin-Singer [10] showed that J_1 is integrable if and only if half of the Weyl curvature tensor of M vanishes (we shall return to this fact in §4). Such twistor spaces provide important examples of complex 3-dimensional manifolds of various algebraic dimensions [90]. Other applications were developed by Bryant [22].

Exercises 2.33 (i) Let $\{e^1, \ldots, e^6\}$ be a basis of 1-forms on $\mathbb{R}^6$, and set

$$\begin{aligned}\Phi &= (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6)\\ &= e^{135} - e^{146} - e^{236} - e^{245} + i(e^{136} + e^{145} + e^{235} - e^{246}).\end{aligned}$$

Show that the almost-complex structure for which Φ is a $(3,0)$-form at each point is integrable if $d\Phi = 0$. Give an example to show that the condition $d(\Re\mathrm{e}\Phi) = 0$ is not sufficient to guarantee integrability.

(ii) By associating to each point of Z_3 its fundamental form, we may regard $Z_3 \cong \mathbb{CP}^3$ as a submanifold of $\bigwedge^2(\mathbb{R}^6)^*$. Describe the intersection of this submanifold with the orthogonal complement of each of the following 2-forms: $e^{12} + e^{34} + e^{56}$, $e^{12} + e^{34}$, and e^{12}.

(iii) Explain the complex quadric interpretation of Z_4 mentioned above. Find out how to define a subbundle (with fibre $\mathbb{CP}^2$) of Z_4 over S^6 that was exploited in [21] to classify pseudo-holomorphic curves in S^6.

3 Compatible connections

In this section, we consider in turn connections preserving g, ω and J, with emphasis on the torsion-free condition. Since g and ω are both bilinear forms, the corresponding theories can be developed in parallel. The study of connections preserving the linear transformation J has more complicated aspects relating to the Nijenhuis tensor of J.

Preliminaries

Let $V \to M$ be a vector bundle. Typically, this will be one of $TM = T$, T^*, $\mathrm{End}\, T = T^* \otimes T$ (which contains the tensors $\mathbb{1}$ and J) or $T^* \otimes T^*$ (which contains g and ω). Let $\Gamma(V)$ denote the space of smooth sections of V over M. Thus, $\Gamma(T^*) = \Omega^1$ is the space of 1-forms over M, and $\Gamma(T) = \mathfrak{X}$ the space of vector fields.

A *connection* on V is an $\mathbb{R}$-linear mapping $\nabla\colon \Gamma(V) \longrightarrow \Gamma(T^* \otimes V)$ such that

$$\nabla(fv) = df \otimes v + f\nabla v, \tag{3.1}$$

whenever v is a section of V and f is a smooth function on M. The section ∇v is called the 'covariant derivative' of v. If $X \in \mathfrak{X}$ and $C_X\colon T^* \otimes V \to V$ is the corresponding contraction then $\nabla_X = C_X \circ \nabla$ satisfies

$$\nabla_X(fv) = (Xf)v + f\nabla_X v.$$

The operator ∇_X is tensorial in X, in the sense that for fixed $v \in \Gamma(V)$ the value $(\nabla_X v)_m$ at a point $m \in M$ depends only on the value X_m.

Let $\nabla, \widetilde{\nabla}$ be two connections on V. To measure their difference, set

$$\widetilde{\nabla}_X = \nabla_X + A_X. \tag{3.2}$$

Since $A_X(fY) = fA_XY$, the element A_X is an endomorphism of V that depends linearly on X. Conversely, given a tensor A with values in $T^* \otimes \mathrm{End}\, V$ at each point, and a connection ∇, then (3.2) satisfies (3.1) and is itself a connection. The space of connections on a vector bundle is therefore an affine space modelled on the vector space $\Gamma(T^* \otimes \mathrm{End}\, V)$.

A connection on the tangent bundle T determines one on T^* by the rule

$$X(\alpha Y) = (\nabla_X \alpha)(Y) + \alpha(\nabla_X Y), \quad \alpha \in \Gamma(T^*). \tag{3.3}$$

Given a connection ∇ on T, and so on T^*, its torsion $\tau = \tau(\nabla)$ may be defined as $d - \wedge \circ \nabla$, where $\wedge$ denotes the mapping $\alpha \otimes \beta \longmapsto \alpha \wedge \beta$. It follows from (3.1) that τ is tensorial, and thus a section of

$$\mathrm{Hom}(T^*, \textstyle\bigwedge^2 T^*) \cong \bigwedge^2 T^* \otimes T \cong \mathrm{Hom}(\bigwedge^2 T, T).$$

By regarding τ as a linear mapping $\bigwedge^2 T \to T$, it is easy to prove the

Lemma 3.4 $\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$.

A connection on T is determined locally by its Christoffel symbols:

$$\nabla_{\partial/\partial x^i}\frac{\partial}{\partial x^j}=\sum_k \Gamma_{ij}^k\frac{\partial}{\partial x^k}$$

or

$$\nabla\frac{\partial}{\partial x^j}=\sum \Gamma_{ij}^k\,dx^i\otimes\frac{\partial}{\partial x^k}.$$

In dual language, this becomes

$$\nabla dx^i=-\sum \Gamma_{jk}^i\,dx^j\otimes dx^k.$$

Thus,

$$\tau(dx^i)=-\sum \Gamma_{jk}^i\,dx^j\wedge dx^k,$$

and ∇ is torsion-free or 'symmetric' iff $\Gamma_{jk}^i=\Gamma_{kj}^i$.

Example 3.5 Let M be a real 2-dimensional submanifold of $\mathbb{R}^3$, parametrized locally by a smooth vector-valued function $\mathbf{r}(p,q)$. The classical first fundamental form $E\,dp^2+2F\,dp\,dq+G\,dq^2$ defined by Gauss is the same thing as the induced Riemannian metric g with components $g_{11}=E$, $g_{12}=F=g_{21}$, $g_{22}=G$ relative to the local coordinates $p=x^1$, $q=x^2$. Let $\mathbf{n}$ be a unit normal vector to M, and $\sum h_{ij}dx^idx^j=Ldp^2+Mdpdq+Ndq^2$ the second fundamental form. The formula

$$\frac{\partial^2\mathbf{r}}{\partial x^i\partial x^j}=\Gamma_{ij}^1\frac{\partial\mathbf{r}}{\partial x^1}+\Gamma_{ij}^2\frac{\partial\mathbf{r}}{\partial x^2}+h_{ij}\mathbf{n}$$

then determines a torsion-free connection on TM that is independent of the choice of coordinates.

Given an arbitrary connection on T, its is easy to check that the one defined by (3.2) with $A_XY=-\frac{1}{2}\tau(X,Y)$ is torsion-free. In terms of Christoffel symbols, this amounts to setting

$$\tilde\Gamma_{jk}^i=\tfrac{1}{2}(\Gamma_{jk}^i+\Gamma_{kj}^i).$$

In this way, a connection on T can be regarded as composed of two pieces, namely its torsion τ and the torsion-free connection $\nabla-\frac{1}{2}\tau$. If $\widetilde\nabla$ and $\widetilde\nabla+A'$ are *both* torsion-free then $A'_XY=A'_YX$, so that (at each point) we may write

$$A'\in\textstyle\bigodot^2T^*\otimes T\subset T^*\otimes T^*\otimes T=T^*\otimes\operatorname{End}T.$$

It follows that mapping $\nabla\longmapsto(\tau,\nabla-\frac{1}{2}\tau)$ is an affine isomorphism mimicking the isomorphism

$$T^*\otimes\operatorname{End}T\cong(\textstyle\bigwedge^2T^*\otimes T)\oplus(\bigodot^2T^*\otimes T)$$

of vector spaces.

We next recall the well-known way in which a connection on a vector bundle V extends to operators on differential forms of all degree with values in V. Let ∇ be a connection on V. Define $\mathbb{R}$-linear operators

$$\nabla_k : \bigwedge^k T^* \otimes V \longrightarrow \bigwedge^{k+1} T^* \otimes V \tag{3.6}$$

for each $k \geqslant 1$ by setting

$$\nabla_k(\alpha \otimes v) = d\alpha \otimes v + (-1)^k \alpha \wedge \nabla v.$$

This gives a sequence

$$\Gamma(T) \xrightarrow{\nabla} \Gamma(T^* \otimes V) \xrightarrow{\nabla_1} \Gamma(\bigwedge^2 T^* \otimes V) \to \cdots \tag{3.7}$$

The *curvature* ρ of ∇ is defined to be the composition

$$\nabla_1 \circ \nabla : \Gamma(V) \to \Gamma(T^* \otimes V) \to \Gamma(\bigwedge^2 T^* \otimes V).$$

The fact that $d^2 = 0$ implies that $\rho(fv) = f\rho(v)$ for $v \in \Gamma(V)$ and f a function. It follows that ρ determines a section of

$$V^* \otimes \bigwedge^2 T^* \otimes V \cong \bigwedge^2 T^* \otimes \operatorname{End} V, \tag{3.8}$$

that we denote by R to avoid confusion with the operator ρ acting on V. The associated curvature operator $R_{XY} \in \operatorname{End} V$ is also defined for any vector fields X, Y, and it may be calculated in the following alternative manner.

Lemma 3.9 $R_{XY} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X.Y]}$.

Proof Follows by writing $\nabla v = \sum \alpha_i \otimes v_i$ for 1-forms α^i. □

One of the most obvious features of the curvature is that if $\varphi \in \Gamma(V)$ satisfies $\nabla\varphi = 0$, then $\rho(\varphi) = 0$, or equivalently $R_{XY}\varphi = 0$ for all X, Y. Typically, V will be some auxiliary vector bundle associated to the manifold (such as $\bigwedge^k T^*$), and ∇ and ρ denote connection and curvature induced from that of the tangent bundle T.

It is easy to show that for any k, the composition

$$\nabla_{k+1} \circ \nabla_k : \bigwedge^k T^* \otimes V \to \bigwedge^{k+2} T^* \otimes V \tag{3.10}$$

is given by $\alpha \otimes v \longmapsto \alpha \wedge \rho(v)$. A connection ∇ is said to be *flat* if its curvature ρ vanishes; in this case, the operators (3.6) give rise to a complex. The vanishing of (3.10) is also the integrability condition for the 'horizontal' distribution D on the total space of V, defined as follows. At a point $v \in \pi^{-1}(m) \in V$, the subspace D_v of the tangent space to V equals $s_*(T_m M)$, where s is any section of V satisfying $s(m) = x$ and $\nabla s|_m = 0$.

Exercises 3.11 (i) Check that (3.3) does define a connection, and define a connection ∇ on $\operatorname{End} T$ with the property that $\nabla \mathbf{1} = 0$.

(ii) Let ∇ be torsion-free, and σ any 2-form. Prove a formula that expresses $d\sigma(X, Y, Z)$ as a universal constant times

$$(\nabla_X \sigma)(Y, Z) + (\nabla_Y \sigma)(Z, X) + (\nabla_Z \sigma)(X, Y).$$

(iii) Let ∇ be a torsion-free connection on the cotangent bundle T^*. Prove the first Bianchi identity, namely that $\wedge \circ \rho = 0$ where $\wedge\colon \bigwedge^2 T^* \otimes T^* \to \bigwedge^3 T^*$.

(iv) A vector field on the total space of V is called *horizontal* if it lies in the distribution D defined directly above. Verify that $\rho = 0$ is equivalent to the condition that the Lie bracket of any two horizontal vector fields is itself horizontal. Deduce the following result.

Theorem 3.12 *Suppose that $\rho = 0$ and $m \in M$. Then on some neighbourhood of m, there exists a basis $\{v_1, \ldots, v_k\}$ of sections of V satisfying $\nabla v_i = 0$ for all i.*

Riemannian and symplectic connections

Now suppose that the metric g is 'covariant constant' relative to a connection ∇ on the tangent bundle T, so that $\nabla g = 0$. This condition involves the natural extension of ∇ to the vector bundle $T^* \otimes T^*$, and means that

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{3.13}$$

for all vector fields X, Y, Z. In local coordinates, with $g = \sum g_{jk} dx^j \otimes dx^k$, the condition becomes

$$\partial_i g_{jk} = \Gamma_{ij}^{\ell} g_{\ell k} + g_{j\ell} \Gamma_{ik}^{\ell} = \Gamma_{ijk} + \Gamma_{ikj}, \tag{3.14}$$

where $\partial_i = \partial / \partial x^i$. From now on, starting with (3.14), we adopt the Einstein convention whereby summation is understood whenever there are repeated indices, one up one down.

Given another connection $\widetilde{\nabla} = \nabla + A$ on T for which $\widetilde{\nabla} g = 0$, set

$$\phi(X, Y, Z) = g(A_X Y, Z).$$

In classical language, we have simply lowered an index of A by setting $\phi_{ijk} = g_{rk} A_{ij}^r$. Then (3.13) implies that $\phi(X, Y, Z) = -\phi(X, Z, Y)$ or $\phi_{ijk} = -\phi_{ikj}$, and

$$\phi \in T^* \otimes \textstyle\bigwedge^2 T^* \subset T^* \otimes T^* \otimes T^*.$$

If ∇ and $\widetilde{\nabla}$ are, in addition, both torsion-free then

$$\phi \in (\textstyle\bigodot^2 T^* \otimes T^*) \cap (T^* \otimes \textstyle\bigwedge^2 T^*) = \ker f, \tag{3.15}$$

where

$$f\colon T^* \otimes \textstyle\bigwedge^2 T^* \subset T^* \otimes T^* \otimes T^* \to \textstyle\bigwedge^2 T^* \otimes T^* \tag{3.16}$$

is the obvious composition. The symmetric group $\mathfrak{S}_3$ acts by permutations on $T^* \otimes T^* \otimes T^*$ and $e = (1,2,3)^3 = ((1,2)(2,3))^3$ acts as both $+1$ and -1 on $\ker f$. Thus f is an isomorphism and $\phi = 0$.

Combined with the remarks following (3.2), the above discussion establishes the existence of the *Riemannian* or *Levi-Civita connection*:

Corollary 3.17 *There exists a unique connection ∇ on T for which $\nabla g = 0$ and $\tau = 0$.*

Let us repeat the above procedure with a non-degenerate 2-form ω in place of g. If ∇ is a torsion-free connection preserving ω then

$$d\omega = \wedge \nabla\omega = 0.$$

Conversely, suppose that $d\omega = 0$. In the notation of (2.4), the connection on U characterized by $\nabla dx^r = 0 = \nabla dy^r$ is torsion-free, and these locally-defined connections can be combined with a partition of unity. It follows that (M,ω) is a symplectic manifold if and only if there exists a torsion-free connection with $\nabla\omega = 0$.

Now suppose that ∇ and $\widetilde{\nabla} = \nabla + A$ are two (not necessarily torsion-free) connections satisfying $\nabla\omega = 0 = \widetilde{\nabla}\omega$, and set

$$\psi(X,Y,Z) = \omega(A_X Y, Z). \tag{3.18}$$

Then $\psi \in T^* \otimes \bigodot^2 T^*$, and the analogue of (3.16) assigns ψ to the difference $\tau(\widetilde{\nabla}) - \tau(\nabla)$ of the torsions. This assignment can be identified with the middle mapping in the naturally-defined exact sequence

$$0 \to \bigodot^3 T^* \to T^* \otimes \bigodot^2 T^* \to \bigwedge^2 T^* \otimes T^* \to \bigwedge^3 T^* \to 0.$$

If $\widetilde{\nabla}, \nabla$ are both torsion-free, then ψ is a section of $\bigodot^3 T^*$.

In the terminology of [47], a *Fedosov manifold* is a symplectic manifold (M,ω) endowed with a choice of torsion-free connection satisfying $\nabla\omega = 0$. In local coordinates, the compatibility condition $\nabla\omega = 0$ is equivalent to

$$\partial_i \omega_{jk} = \Gamma_{ij}^{\ell} \omega_{\ell k} - \omega_{j\ell} \Gamma_{ik}^{\ell}.$$

If we modify the Christoffel symbols by setting $\Gamma_{ijk} = \Gamma_{ij}^r \omega_{rk}$ (in contrast to the more conventional meaning (3.14)), then Γ_{ijk} is totally symmetric. Note that in the coordinates of (2.4), the only non-zero values of ω_{rk} are ± 1.

The above situation can be summarized by the following diagram, in which $\sim$ denotes affine isomorphism. It is valid on any manifold equipped with a metric g and a non-degenerate 2-form ω, not necessarily compatible. The less familiar lower mapping $\sim$ is described in (iii) below, and [47] raises the question of computing the circular composition.

$$\begin{tikzcd}
 & g\text{-connections} \arrow[d] & \\
\begin{array}{c}\text{torsion}\\ \text{tensors}\end{array} \arrow[ur, "\sim"] & \begin{array}{c}\text{all}\\ \text{connections}\end{array} \arrow[l] \arrow[r] & \begin{array}{c}\text{torsion-free}\\ \text{connections}\end{array} \arrow[dl, "\sim"] \\
 & \omega\text{-connections} \arrow[u] &
\end{tikzcd}$$

Exercises 3.19 (i) Prove that the connection defined in Example 3.5 coincides with the Levi-Civita connection of the induced metric g on the surface M.

(ii) If ∇^g, ∇^h denote the Levi-Civita connections for the metrics related by (2.23), try to find an expression for the difference tensor $A_X = \nabla^g_X - \nabla^h_X$.

(iii) Let $\Delta_{ijk} = \Delta_{ij}^m \omega_{mk}$ be the Christoffel symbols of a torsion-free connection on a manifold equipped with a non-degenerate 2-form ω. Show that

$$\Gamma_{ijk} = (\Delta_{ijk} + \Delta_{ikj} - \Delta_{jki}) + \tfrac{1}{2}(\partial_k \omega_{ij} - \partial_i \omega_{jk} - \partial_j w_{ki})$$

are the Christoffel symbols of a connection preserving ω. Simplify the last bracket under the symplectic assumption $dw = 0$.

Derivatives of J

Suppose that (M, J) is an almost-complex manifold. Choose any torsion-free connection ∇ on (the tangent bundle of) M. Using (1.17) and Lemma 3.4, this allows us to write

$$N(X, Y) = \sigma_X Y - \sigma_Y X,$$

where

$$\begin{aligned} \sigma_X Y &= \nabla_{JX} JY - \nabla_X Y - J\nabla_{JX} Y - J\nabla_X JY \\ &= (\nabla_{JX} J)Y - J(\nabla_X J)Y, \end{aligned}$$

and given that $\nabla_X (J \circ J) = 0$,

$$\sigma_X = \nabla_{JX} J + (\nabla_X J)J.$$

From the analogous property of N (1.18), we see that σ is the tensor characterized by

$$\sigma_X Y = -8\,\Re\mathrm{e}\,(\nabla_{X^{1,0}} Y^{1,0})^{0,1}.$$

Thus, $\sigma = 0$ if and only if $\nabla_A B \in T^{1,0}$ for all vector fields A, B of type $(1,0)$.

Corollary 3.20 *If ∇ is a torsion-free connection such that $\nabla J = 0$ then J is integrable.*

Conversely, if J is integrable there exists a torsion-free connection such that $\nabla J=0$. This follows from patching together locally-defined connections for which $\nabla dx^i=0=\nabla dy^j$ relative to the coordinates of (1.11).

We can improve the corollary by replacing $\nabla J=0$ by the condition $\nabla_1 J=0$, where ∇_1 is defined as in (3.7) with $V=\operatorname{End}T=T^*\otimes T$. Whilst ∇J has values in $T^*\otimes T^*\otimes T$, the derivative $\nabla_1 J$ is essentially its skew component in $\bigwedge^2 T^*\otimes T$, assuming always that ∇ is torsion-free. To see that Corollary 3.20 remains valid with the new hypothesis, we first use the metric to define

$$\Phi(X,Y,Z)=g((\nabla_X J)Y,Z). \tag{3.21}$$

Observe that

$$g(\sigma_X Y,Z)=\Phi(JX,Y,Z)+\Phi(X,JY,Z),$$

and that to obtain $g(N(X,Y),Z)$ we need to skew the right-hand side over X and Y. The result of doing this is however the same as first skewing in the first two positions of Φ. Thus,

$$g(N(X,Y),Z)=(\wedge\circ\Phi)(JX,Y,Z)+(\wedge\circ\Phi)(X,JY,Z), \tag{3.22}$$

where $\wedge\colon T^*\otimes T^*\otimes T^*\to\bigwedge^2 T^*\otimes T^*$ is the anti-symmetrization. The condition $\nabla_1 J=0$ implies that $\wedge\circ\Phi=0$, so that $N=0$.

Lemma 3.23 *If ∇ is the Levi-Civita connection then*

$$\Phi(X,Y,Z)=-\Phi(X,Z,Y)=-\Phi(X,JY,JZ).$$

Proof The first inequality follows because J, and so $\nabla_X J$, is skew-adjoint relative to g. The second since $\nabla_X J\circ J=-J\circ\nabla_X J$. □

For the remainder of Section 3, ∇ denotes the Levi-Civita connection. Lemma (3.23) implies that

$$\Phi\in T^*\otimes[\![\Lambda^{2,0}]\!], \tag{3.24}$$

reflecting the fact that $\nabla_X\omega\in[\![\Lambda^{2,0}]\!]$ for all X. There are injective mappings

$$\begin{aligned}\alpha&\colon\Lambda^{3,0}\to\Lambda^{1,0}\otimes\Lambda^{2,0},\\ \beta&\colon\Lambda^{1,0}\to\Lambda^{2,1},\end{aligned}$$

given by $\alpha(A\wedge B\wedge C)=A\otimes(B\wedge C)+B\otimes(C\wedge A)+C\otimes(A\wedge B)$ and $\beta(A)=A\wedge\omega$, where A,B,C are (1,0)-forms and ω is the fundamental 2-form. Let $V=\ker\alpha^*$, and $\Lambda_0^{2,1}=\ker\beta^*$, where the asterisk denotes adjoint with respect to the natural Hermitian metric. Then V and $\Lambda_0^{2,1}$ are known to be irreducible under $U(n)$. It follows that the space in (3.24) has 4 components under the action of $U(n)$ for $n\geqslant 3$, and leads to the characterization in [53] of $2^4=16$ classes of almost-Hermitian manifolds. For example, the $(1,2)$-component of Φ can be identified with $(d\omega)^{1,2}$, and the vanishing of this is exactly complementary to the condition that $N=0$.

Corollary 3.25 *M is Kähler if and only if $\nabla J = 0$.*

The condition $\nabla J = 0$ is equivalent to asserting that the holonomy group is contained in $U(n)$.

Suppose that M is Hermitian so that $N = 0$. The remark after (3.16), together with (3.22), implies that $(A,B,C) \longmapsto \Phi(A,B,C)$ is zero for all $A,B,C \in T^{1,0}$. Also $\Phi(A,B,\overline{C})$, so $\sigma = 0$.

Corollary 3.26 *The following are equivalent:*
(i) *J is integrable,*
(ii) $\nabla_{JX} J = J(\nabla_X J)$,
(iii) $A, B \in \Gamma(T^{1,0}) \Rightarrow \nabla_A B \in \Gamma(T^{1,0})$.

Condition (iii) has important consequences for the curvature tensor.

Exercises 3.27 (i) Let $\Lambda_0^{1,1}$ denote the orthogonal complement of ω in $\Lambda^{1,1}$. Show that as a $U(n)$-module it can be identified with the Lie algebra $\mathfrak{su}(n)$ (and adjoint representation), and that $\Lambda_0^{1,1}$ is irreducible.

(ii) Let M be an almost-Hermitian manifold of real dimension $2n \geqslant 6$. Deduce that Φ lies in a real vector space that is the direct sum of four $U(n)$-invariant real subspaces of dimensions $2d$, $n^3 - n^2 - 2d$, $2n$, $n^3 - n^2 - 2n$, where $d = \binom{n}{3}$. What happens when $n = 2$?

(iii) Let M be a 6-dimensional almost-Hermitian manifold. A sequence of differential operators

$$0 \to \Gamma(\Lambda^{0,0}) \to \Gamma(\Lambda^{1,0} \oplus \Lambda^{0,1}) \to \Gamma(\Lambda_0^{1,1} \oplus \Lambda^{2,0}) \to \Gamma(\Lambda_0^{2,1}) \to 0$$

is defined by composing d with the appropriate projection. Find necessary and sufficient conditions on Φ that guarantee that this is a complex [92].

The Riemann curvature tensor

Let M be a Riemannian manifold of dimension $2n$, and ∇ its Levi-Civita connection. It follows from the fact that $\nabla_X g = 0$ that the operator R_{XY} is a skew-adjoint transformation of the inner product space (T, g). The tensor

$$R(X,Y,Z,W) = g(R_{XY}Z, W)$$

is therefore skew not just in X, Y, but also in Z, W. Let $\{e_i\}$ be an orthonormal basis of vector fields, and set $R_{ijkl} = R(e_i, e_j, e_k, e_l)$, so that

$$R_{ijkl} = R_{jilk}. \tag{3.28}$$

The fact that ∇ is torsion-free gives the Bianchi identity

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0 \tag{3.29}$$

(see Exercises (3.11)), with the well-known consequence

Lemma 3.30 $R_{ijkl} = R_{klij}$.

Proof This can be reformulated in terms of the alternating group $\mathfrak{A}_4$ that acts on $\bigotimes^4 T^*$ by permuting the factors. Let R be an element of $\bigotimes^4 T^*$ satisfying (3.28) and (3.29), so that it makes sense to consider the span $\mathbb{R}\mathfrak{A}_4 \cdot R$ of the orbit of R. Let B be the subgroup of $\mathfrak{A}_4$ generated by the double transposition $(12)(34)$, C that generated by the cycle (123), and $e \in \mathfrak{A}_4$ the identity. Then the 2 elements in any 'coset' $\sigma B \cdot R$ are equal, and the 3 elements in $\tau C \cdot R$ add to zero. Since $|\sigma B \cap \tau C| \leqslant 1$ and $|BC \cap CB| = 4$, it follows that

$$e + (12)(34) - (13)(42) - (14)(23) \in \mathbb{R}\mathfrak{A}_4$$

annihilates R. Thus, $(13)(42) \cdot R = R$, as required. □

The lemma implies that $\bigwedge^2(\bigwedge^2 T^*)$ injects into $T^* \otimes \bigwedge^3 T^*$, and R belongs to the space

$$\mathcal{R} = \ker\left(\bigodot^2(\bigwedge^2 T^*) \longrightarrow \bigwedge^4 T^*\right).$$

It is well known that $\mathcal{R}$ consists of three irreducible components under the action of $O(2n)$ for $n \geqslant 2$ (see Exercises 3.11), and these are described classically as follows. The Ricci tensor is defined by

$$R_{il} = R_{ijkl}\, g^{jk},$$

and represents a contraction $\mathcal{R} \to \bigodot^2 T^*$. Its 'trace' $s = R_{il} g^{il}$, obtained by further contraction $\bigodot^2 T^* \to \mathbb{R}$, is by definition the scalar curvature. The manifold is *Einstein* if the Ricci tensor is proportional to the metric, and this forces $R_{il} = (s/n) g_{il}$. The second Bianchi identity can be used to prove that in this case s is constant, still assuming that $n \geqslant 2$ [51].

Let

$$\begin{aligned} A_{ijkl} &= R_{jk} g_{il} - R_{jl} g_{ik} - R_{ik} g_{jl} + R_{il} g_{jk}, \\ B_{ijkl} &= s(g_{jk} g_{il} - g_{jl} g_{ik}). \end{aligned}$$

Recalling that the real dimension equals $2n$, we seek functions $a(n), b(n)$ such that

$$R_{ijkl} = W_{ijkl} + a(n) A_{ijkl} + b(n) B_{ijkl}, \quad W_{ijkl} g^{jk} = 0.$$

This will ensure that W represents the Weyl tensor, by definition that part of the curvature with zero Ricci contraction. The second equation implies that $1/a(n) = 2(n-1)$ and $1/b(n) = -2(n-1)(2n-1)$. Since

$$A_{ijkl} g^{jk} g^{il} = 2(2n-1)s, \quad B_{ijkl} g^{jk} g^{il} = 2n(2n-1)s,$$

it also follows that

$$R_{ijkl} = W_{ijkl} + C_{ijkl} + \frac{1}{2n(2n-1)} B_{ijkl},$$

where $C_{ijkl} = (A_{ijkl} - \frac{1}{n} B_{ijkl})/(2n-2)$ represents the tracefree Ricci tensor.

Exercises 3.31 (i) Let T be a real inner product space of dimension $2n \geqslant 4$. Relative to $O(2n)$, there are equivariant isomorphisms

$$\begin{aligned} \bigodot^2(\bigwedge^2 T^*) &\cong \mathbb{R} \oplus \bigodot_0^2 T^* \oplus \bigwedge^4 T^* \oplus \mathcal{W}, \\ \bigwedge^2(\bigwedge^2 T^*) &\cong \bigwedge^2 T^* \oplus \mathcal{X}, \\ T^* \otimes \bigwedge^3 T^* &\cong \bigwedge^4 T^* \oplus \bigwedge^2 T^* \oplus \mathcal{Y}, \end{aligned}$$

where all the summands are irreducible. Assuming this, compute the dimensions of $\mathcal{X}, \mathcal{Y}, \mathcal{W}$, and show that they all vanish if $n = 3$. Using Schur's lemma (the elementary fact that any G-homomorphism between irreducible spaces is either zero or an isomorphism), show that $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic, and $\mathcal{W}$ lies in the kernel of the skewing map $\bigodot^2(\bigwedge^2 T^*) \to \bigwedge^4 T^*$.

(ii) Let M be an Einstein manifold. Its curvature tensor R may be regarded as an element of $\mathbb{R} \oplus \mathcal{W}$ at each point. Show that the tensor $S_{lm} = R^{ij}{}_{kl} R^k{}_{ijm}$ is determined by a linear mapping $\bigodot^2 \mathcal{W} \to T^* \otimes T^*$. Deduce from Section 4 that if $\dim M = 4$ then S_{lm} is a scalar multiple of g_{lm} (this is called the 'super-Einstein' condition).

Now let (M, g, J) be an almost-Hermitian manifold. In arbitrary dimension, the type decomposition (2.28) induces a decomposition of real vector spaces

$$\bigodot^2(\bigwedge^2 T^*) = \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{S}_3 \oplus \mathcal{S}_4,$$

where

$$\begin{aligned} \mathcal{S}_1 &= [\Lambda^{1,1} \odot \Lambda^{1,1}], \\ \mathcal{S}_2 &= [\Lambda^{2,0} \odot \Lambda^{0,2}], \\ \mathcal{S}_3 &= [\![\Lambda^{2,0} \odot \Lambda^{2,0}]\!], \\ \mathcal{S}_4 &= [\![\Lambda^{2,0} \odot \Lambda^{1,1}]\!]. \end{aligned}$$

(If U_1, U_2 are subspaces of V then $U_1 \odot U_2$ denotes the image of $U_1 \otimes U_2$ under the symmetrization $V \otimes V \to \bigodot^2 V$. Addition of complex conjugates is understood when there are double brackets, in accordance with (1.16).) For example, $R \in \bigodot^2(\bigwedge^2 T^*)$ belongs to the subspace $\mathcal{S}_1$ if and only if

$$R(X, Y, Z, W) = R(X, Y, JZ, JW) \tag{3.32}$$

for all tangent vectors X, Y, Z, W.

Lemma 3.33 *If M is Kähler then its Riemann tensor R belongs to the space $\mathcal{R}_1 = \mathcal{R} \cap \mathcal{S}_1$.*

Proof The Levi-Civita connection ∇ induces a connection on $\operatorname{End} T$, whose curvature is induced from the natural Lie algebra action of $\operatorname{End} T$ on $\operatorname{End} T$. Thus, if R_{XY} is the operator of Lemma 3.9 then

$$0 = (\nabla_1 \nabla J)_{XY} = R_{XY}(J) = R_{XY} \circ J - J \circ R_{XY}. \tag{3.34}$$

When we convert R into a tensor with all lower indices, the relation $R_{XY} \circ J = J \circ R_{XY}$ translates into the equation (3.32). $\square$

Define $\mathcal{R}_2$ so that the right-hand side of

$$\mathcal{R} \cap (\mathcal{S}_1 \oplus \mathcal{S}_2) = \mathcal{R}_1 \oplus \mathcal{R}_2$$

is an orthogonal sum, and set

$$\mathcal{R}_3 = \mathcal{R} \cap \mathcal{S}_3, \qquad \mathcal{R}_4 = \mathcal{R} \cap \mathcal{S}_4.$$

Since the images of $\mathcal{S}_1 \oplus \mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}_4$ in $\bigwedge^4 T^*$ are mutually orthogonal, it follows that

$$\mathcal{R} = \bigoplus_{i=1}^{4} \mathcal{R}_i.$$

This notation is consistent with the spaces $\mathcal{L}_j = \bigoplus_{i=1}^{j} \mathcal{R}_i$, $j = 1, 2, 3$, defined by Gray [52], although a full analysis of the $U(n)$-components of $\mathcal{R}$ was subsequently carried out by Tricerri and Vanhecke [102; 38].

Lemma 3.35 *If M is Hermitian then $R \in \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \mathcal{R}_4$.*

Proof This follows from Corollary 3.26. Suppose that J is integrable, and let A, B, C be vector fields of type $(1,0)$. Then

$$R_{AB}C = \nabla_A \nabla_B C - \nabla_B \nabla_A C - \nabla_{[A,B]} C$$

also has type $(1,0)$, and so $g(R_{AB}C, D) = 0$ for all $A, B, C, D \in T^{1,0}$. Put another way, R has no component in $\Lambda^{2,0} \otimes \Lambda^{2,0}$ nor (taking complex conjugates) in $\Lambda^{0,2} \otimes \Lambda^{0,2}$. The result follows from the definition of $\mathcal{R}_3$. $\square$

If $R \in \mathcal{R}_3$ then $R = \sigma + \overline{\sigma}$ where $\sigma \in \Lambda^{2,0} \otimes \Lambda^{2,0}$. Since $\Lambda^{2,0}$ is isotropic, the Ricci contraction annihilates σ (and similarly $\overline{\sigma}$). Thus, $\mathcal{R}_3$ is a component of the subspace $\mathcal{W}$ of $\mathcal{R}$ generated by Weyl tensors. A dimension count shows that $\dim \mathcal{R}_3 > \frac{1}{8} \dim \mathcal{W}$ for $n \geqslant 2$ [93], and this gives some idea of how the existence of a single OCS conditions the Weyl tensor.

Let R_i denote the component of the Riemann tensor R in $\mathcal{R}_i$. The following result, taken from [38], highlights the fundamental nature of the space $\mathcal{R}_1$ of Kähler curvature tensors.

Proposition 3.36 *The tensors R_2, R_3, R_4 are linear functions of $\nabla^2 J$.*

Proof The kernel of the mapping $R \longmapsto R(J)$ of (3.34) equals $\mathcal{R}_1$, and $R(J)$ can be identified with $R_2 \oplus R_3 \oplus R_4$. $\square$

Let M be a Kähler manifold of real dimension $2n$, and let $\kappa = \Lambda^{n,0}$ denote its *canonical line bundle*. Let ξ be a local section of κ, so that the Levi-Civita connection satisfies

$$\nabla \xi = i\alpha \otimes \xi \tag{3.37}$$

for some *real* 1-form α. The curvature of κ is given by

$$\rho(\xi) = \nabla_1 \nabla \xi = i\, d\alpha \wedge \xi,$$

and is, to all intents and purposes, the same as the closed 2-form $\Omega = id\alpha$. The latter will not in general be globally exact, since α is only defined locally.

Algebraically, Ω can be identified with the image of R under the contraction

$$\bigwedge^2 T^* \otimes \operatorname{End} T \to \bigwedge^2 T^* \otimes (\operatorname{End} \kappa) \cong \bigwedge^2 T^*, \tag{3.38}$$

given in lowered index notation by

$$R_{ijkl} \longmapsto R_{ijkl}\omega^{kl} = -(R_{iklj} + R_{iljk})\omega^{kl} = 2R_{kilj}\omega^{kl},$$

where ω is the fundamental 2-form. It follows that

$$\Omega(X,Y) = 2S(JX,Y)$$

is (twice) the *Ricci form*, a $(1,1)$-form manufactured in the natural way from the Ricci tensor, just as ω is from g.

Example 3.39 Further to Example 2.30, the canonical bundle is generated by $\eta = \omega_2 + i\omega_3$. To determine α in (3.37), note that $i\alpha \wedge \eta = d\eta = -3e^4 \wedge \eta$. Thus $(i\alpha + 3e^4) \wedge \eta = 0$, so $i\alpha + 3e^4$ must be a $(1,0)$-form and $\alpha = -3e^3$. Hence, $\Omega = -6i\omega_1$, showing that g_2 is Einstein.

We conclude this section by discussing very briefly properties of the curvature of a torsion-free symplectic connection. Let M be a manifold of dimension $2n \geqslant 4$ with a symplectic form ω, and let ∇ be a connection on M satisfying $\tau = 0$ and $\nabla\omega = 0$. The curvature of the induced connection on T^* is a linear mapping $\rho\colon T^* \to \bigwedge^2 T^* \otimes T^*$, and we define

$$R_{ijkl} = \omega_{ir}\rho(e^r)(e_k, e_l, e_j),$$

with summation over r. It follows easily that $R_{ijkl} = R_{jikl}$, whence

$$R \in \ker\left(\bigodot^2 T^* \otimes \bigwedge^2 T^* \longrightarrow T^* \otimes \bigwedge^3 T^*\right). \tag{3.40}$$

Exterior powers of T^* are not irreducible for $Sp(2n,\mathbb{R})$. For example, $\bigwedge^2 T^*$ contains the trivial 1-dimensional space $\langle\omega\rangle$ generated by the symplectic form ω; more generally wedging with ω determines an equivariant mapping $\bigwedge^k T^* \to \bigwedge^{k+2} T^*$. In this sense, the situation is dual to that of the orthogonal group, and symmetric powers of T^* *are* irreducible for $Sp(2n,\mathbb{R})$. It follows that the kernel in (3.40) contains only two $Sp(2n,\mathbb{R})$-irreducible summands. The conclusion is that there is a unique 'Ricci tensor' in the symplectic situation, but no way of defining scalar curvature [103].

Recent problems associated to the theory of symplectic connections can be found in [27] and references therein.

Exercises 3.41 (i) Consider the manifold X of Example 1.27. Verify that $\omega = e^1 \wedge e^4 + e^2 \wedge e^3$ is closed. Determine possible constants τ^i_j for which $\nabla e^i = \sum_j \tau^i_j \otimes e^j$ defines a torsion-free connection on TX for which $\nabla\omega = 0$. Compute the curvature of this connection.

(ii) Let M be a hyperkähler manifold (Definition 2.10). Its holomorphic cotangent space $E = \Lambda^{1,0}$ has a quaternionic structure, and an even tensor product such as $\bigodot^4 E$ is the complexification of a real space $[\bigodot^4 E]$. By referring to [92] or [60], explain why the curvature tensor of M may be regarded as an element of $[\bigodot^4 E]$ at each point.

4 Further topics

This section is devoted to applications of the preceding ones, and is divided into three subsections. The first two are concerned with four-dimensional Riemannian geometry, and the third with the theory of special Kähler manifolds.

By way of preliminaries, we specialize the discussion of the curvature of the Levi-Civita connection on an almost-Hermitian manifold in Section 3 to the case of four real dimensions. Orientation plays an important role, and attention is focussed on the semi Weyl tensor W_+. This leads to a summary of well-known relations with topology. In the second subsection, results are applied to give a crude classification of 4-dimensional Riemannian manifolds, distinguished according to the number of orthogonal complex structures that exist locally. This is rounded off by a description of constraints arising from the decomposition of W_+ in the context of almost-Kähler manifolds.

In highlighting the role of curvature, it seems only right to include a situation in which the curvature is zero. Many interesting integrability conditions can be interpreted by the vanishing of the curvature of some connection or family of connections. Such situations are playing an increasingly important role in differential geometry, and we illustrate one involving a class of flat symplectic connections. This is used in the definition of special Kähler metrics and associated hyperkähler metrics.

Curvature in four dimensions

Let M be an almost-Hermitian manifold of real dimension 4, and let T denote the tangent space at an arbitrary point. Forgetting about J for the moment, the curvature operator $\hat{R}$ is an endomorphism of

$$\textstyle\bigwedge^2 T^* = \Lambda^+ \oplus \Lambda^-. \tag{4.1}$$

It therefore decomposes under the action of $SO(4)$ as a block matrix

$$\hat{R} = \begin{pmatrix} A_+ & B \\ B^T & A_- \end{pmatrix},$$

where $A_\pm$ is a symmetric matrix, corresponding to a self-adjoint endomorphism of $\Lambda^\pm$. For example, regarding $\hat\omega_1^\pm = (e^{12} \pm e^{34})/\sqrt{2}$ as unit 2-forms, we see that

$$\begin{array}{rcl} (A_+)_{11} &=& R(\hat\omega_1^+, \hat\omega_1^+) = \frac{1}{2}(R_{1212} + 2R_{1234} + R_{3434}),\\ B_{11} &=& R(\hat\omega_1^+, \hat\omega_1^-) = \frac{1}{2}(R_{1212} - 2R_{1234} + R_{3434}). \end{array} \tag{4.2}$$

Hence, using (3.29),

$$\operatorname{tr} A_+ = \operatorname{tr} A_- = \tfrac{1}{4}\sum_{i,j} R_{ijij} = \tfrac{1}{4}s,$$

where s is the scalar curvature.

The traceless endomorphisms $W_\pm = A_\pm - \frac{1}{12}s\mathbb{1}$ represent two halves of the Weyl tensor, whereas $B \in \operatorname{Hom}(\Lambda^-, \Lambda^+) \cong \bigodot^2_0 T^*$ represents the trace-free part of the Ricci tensor. The last isomorphism is obtained by contracting on the middle two indices as in the example

$$\begin{array}{rcl} 2\hat\omega_1^+ \otimes \hat\omega_2^- &=& (e^{12} + e^{34}) \otimes (e^{13} - e^{42})\\ &=& (e^1e^2 - e^2e^1 + e^3e^4 - e^4e^3)(e^1e^3 - e^3e^1 - e^4e^2 + e^2e^4)\\ &\longmapsto& (e^1e^4 - e^2e^3 - e^3e^2 + e^4e^1) = (e^1 \odot e^4 - e^2 \odot e^3). \end{array}$$

Symbolically, we may now write

$$R = (W_+, W_-, B, s) \qquad [5,5,9,1], \tag{4.3}$$

where the numbers in square brackets indicate the dimension of the corresponding space of tensors. It follows that M is Einstein if and only if B is identically zero, and M is conformally flat if and only if $W_+ \equiv 0 \equiv W_-$.

Definition 4.4 *M is half conformally flat if either W_+ or W_- vanishes identically. In the latter case, M is called self-dual, and in the former case anti-self-dual or ASD.*

Now suppose that M is almost-Hermitian, and recall that J reduces Λ^+ into the sum $\langle\omega\rangle \oplus [\![\Lambda^{2,0}]\!]$ of real subspaces of dimension 1 and 2. Relative to this, we have

Lemma 4.5 *$\hat R$ decomposes as a symmetric matrix*

$$\left(\begin{array}{c|c||c} \frac{1}{4}s^* & W' & B' \\ \hline \bullet & W'' + \frac{1}{8}(s - s^*) & B'' \\ \hline\hline \bullet & \bullet & W_- + \frac{1}{12}s \end{array}\right) \tag{4.6}$$

with $\operatorname{tr}(W'') = 0$.

The effect of J is therefore to refine (4.3) by the additional splittings

$$\begin{array}{ll} B = (B', B'') & [3,6],\\ W_+ = (s - 3s^*, W', W'') & [1,2,2], \end{array}$$

with the indicated dimensions. The metric is said to be $*$*Einstein* if $B' = 0$ and $W' = 0$, and *strongly* $*$*Einstein* if, in addition, s^* is constant.

Lemma 4.7 (i) *If M is Kähler, then $W', W'', s-s^*, B''$ all vanish.*
(ii) *If M is Hermitian then W'' vanishes.*

Proof This is a re-interpretation of Lemmas 3.33 and 3.35. The Kähler condition implies that R is effectively an endomorphism of $\Lambda^{1,1}$. It follows that all tensors above involving $[\![\Lambda^{2,0}]\!]$ vanish. Put another way, one eliminates all the components corresponding to a big cross $+$ in the matrix (4.6), and (i) follows. If J is integrable, we have seen that R has no component in

$$\Lambda^{2,0} \otimes \Lambda^{2,0} \cong \mathrm{Hom}(\Lambda^{0,2} \otimes \Lambda^{2,0}).$$

The latter corresponds to the trace-free part of the real space $\mathrm{Hom}([\![\Lambda^{2,0}]\!], [\![\Lambda^{2,0}]\!])$, and is represented by W''. □

In the Kähler case, we may therefore write

$$R = (W_-, B', s) \qquad [5, 3, 1],$$

and M is ASD if and only if $s = 0$. Kähler metrics satisfying $s \equiv 0$ are called 'scalar-flat Kähler' or SFK, and their study forms part of the more general theory ofextremal Kähler metrics introduced in [29]. The tensor W_- is the so-called Bochner tensor that in higher dimensions is generalized by a corresponding component of the space $\mathcal{R}_1$ of Lemma 3.33 [109]. Self-dual Kähler surfaces are studied in [34; 4; 23]. Finally, we remark that an almost-Hermitian 4-manifold satisfying $W'' = 0$ is said to have 'Hermitian Weyl tensor'.

We have already seen that on a Kähler manifold, the Ricci tensor can be extracted as the curvature of the canonical bundle. In the case of an arbitrary 4-dimensional almost-Hermitian manifold, we can easily distinguish the components of R that contribute to the curvature 2-form Ω of the canonical bundle $\kappa = \Lambda^{2,0}$. The contraction (3.38) amounts to selecting the first row of (4.6) and, identifying W', B' with 2-forms,

$$\Omega = \tfrac{1}{4} s^* \omega + W' + B', \qquad W' \in [\![\Lambda^{2,0}]\!] \subset \Lambda^+,\ B' \in \Lambda^-. \tag{4.8}$$

Corollary 4.9 *The induced connection on the canonical line bundle is self-dual if M is Einstein, and ASD if M is SFK.*

The forms in (4.8) may be used to represent the first Chern class

$$c_1(T^{1,0}) = c^+ + c^- \in H^+ \oplus H^- \tag{4.10}$$

of M, where $H^\pm = \{\alpha \in \Gamma(\Lambda^\pm) : d\alpha = 0\}$. The corollary is in theory relevant to constructions of self-dual metrics by Joyce and others, described by [62; 31] and references therein.

Let M be a connected compact oriented 4-manifold M. Its Betti numbers are defined by $b_i = \dim H^i(M, \mathbb{R})$, and satisfy $b_0 = 1 = b_4$ and $b_1 = b_3$ by

Poincaré duality. The Euler characteristic is then

$$\chi = \sum_{i=0}^{4} (-1)^i b_i = 2 - 2b_1 + b_2,$$

and is positive if M is simply-connected. Applying Hodge theory to the decomposition (4.1) yields $b_2 = b^+ + b^-$, where $b^\pm = \dim H^\pm$. The quantity

$$\sigma = b^+ - b^-$$

is the signature of the real quadratic form associated to the intersection form

$$H_2(M,\mathbb{Z}) \times H_2(M,\mathbb{Z}) \to H_4(M,\mathbb{Z}) \cong \mathbb{Z}. \tag{4.11}$$

on homology. We now explain how both quantities χ, σ can be computed from a knowledge of the curvature tensor R of the Levi-Civita connection.

Recall that R is matrix of 2-forms ($\rho_{ij} = \sum R_{ijkl} e^k \wedge e^l$). Invariant polynomials on the Lie algebra $\mathfrak{so}(4)$, isomorphic to the space (4.1), provide the following antisymmetric matrices of 2-forms:

$$\rho = \begin{pmatrix} 0 & \rho_{12} & \rho_{13} & \rho_{14} \\ \bullet & 0 & \rho_{23} & \rho_{24} \\ \bullet & \bullet & 0 & \rho_{34} \\ \bullet & \bullet & \bullet & 0 \end{pmatrix}, \quad *\rho = \begin{pmatrix} 0 & \rho_{34} & -\rho_{24} & \rho_{23} \\ \bullet & 0 & \rho_{14} & -\rho_{13} \\ \bullet & \bullet & 0 & \rho_{12} \\ \bullet & \bullet & \bullet & 0 \end{pmatrix}.$$

Here, $*$ can be regarded as the involution acting as ± 1 on $\Lambda^\pm$. In terms of these matrices,

$$\begin{aligned} -\operatorname{tr}(\rho^2) &= \sum_{i,j} {\rho_{ij}}^2, \\ \tfrac{1}{4}\operatorname{tr}(\rho(*\rho)) &= \rho_{12}\rho_{34} + \rho_{13}\rho_{42} + \rho_{14}\rho_{23}. \end{aligned}$$

The latter is the so-called Pfaffian, formally the square root of $\det \rho$.

The Hirzebruch signature theorem implies that σ equals $\frac{1}{3}p_1$, where p_1 is the first Pontrjagin number. The latter can be computed by Chern-Weil theory as

$$\begin{aligned} p_1 &= -\tfrac{1}{8\pi^2} \int \operatorname{tr}(\rho \wedge \rho) \\ &= \tfrac{1}{4\pi^2} \int \Big((|A_+|^2 + |B|^2) - (|B^T|^2 + |A_-|^2) \Big) \upsilon \\ &= \tfrac{1}{4\pi^2} \int \Big(|W_+|^2 - |W_-|^2 \Big) \upsilon. \end{aligned}$$

Thus if M is ASD then $\sigma \leqslant 0$, with equality if M is conformally flat. On the other hand, by Chern's theorem,

$$\begin{aligned} \chi &= \tfrac{1}{32\pi^2} \int \operatorname{tr}(\rho \wedge *\rho) \\ &= \tfrac{1}{8\pi^2} \int \Big((|A_+|^2 - |B|^2) - (|B^T|^2 - |A_-|^2) \Big) \upsilon \\ &= \tfrac{1}{8\pi^2} \int \Big(|W_+|^2 + |W_-|^2 + 6(\tfrac{1}{12}s)^2 - 2|B|^2 \Big) \upsilon. \end{aligned}$$

Thus, if $B=0$,

$$\chi = \tfrac{2}{3}|\sigma| + \frac{1}{8\pi^2}\int \Big(2|W_\pm|^2 + \tfrac{1}{24}s^2\Big)\upsilon.$$

This provides the celebrated Hitchin-Thorpe inequality:

Corollary 4.12 *An Einstein 4-manifold satisfies* $\chi \geqslant \frac{3}{2}|\sigma|$, *with equality iff* $s=0$ *and* $W_\pm = 0$.

If M is almost-complex then there are corresponding formulae for the indices of the Dolbeault complexes discussed in Section 2. In particular, the *arithmetic index* $1-h^{0,1}+h^{0,2}$ equals

$$\tfrac{1}{12}(c_1^2+c_2) = \tfrac{1}{12}(p_1+3\chi) = \tfrac{1}{4}(\chi+\sigma).$$

So a compact 4-manifold can only have an almost-Hermitian structure if

$$\chi+\sigma \equiv 0 \bmod 4.$$

Conversely, if the second Stiefel-Whitney class $w_2(M)$ is the reduction mod 2 of $c\in H^2(M,\mathbb{Z})$ and $c^2 = 2\chi+3\sigma$ then M admits an almost-complex structure J for which $c=c_1$. If M is simply-connected, such a c exists in H^+ if and only if b^+ is odd.

Example 4.13 The complex projective plane $\mathbb{CP}^2$ has

$$p_1 = {c_1}^2 - 2c_2 = (3x)^2 - 2(3x^2) = 3x^2,$$

where $x = c_1(L)$ is the positive generator of $H^2(\mathbb{CP}^2,\mathbb{Z})$ (and L denotes the standard holomorphic line bundle), which is consistent with the fact that $\chi=3$ and $\sigma=1$. More generally, one may consider the connected sum

$$m\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2 \tag{4.14}$$

of m copies of $\mathbb{CP}^2$ and n copies of the same smooth manifold with reversed orientation. The result has signature $\sigma = m-n$ and Euler characteristic $\chi = 2+m+n$. If $m\geqslant n$, one requires $m \leqslant 4+5n$ for the possibility of an Einstein metric. Equality is not possible for the following reason. If

$$R = \begin{pmatrix} 0 & 0 \\ 0 & W_- \end{pmatrix} \tag{4.15}$$

then R has no component in $\bigwedge^2 T^*\otimes \operatorname{End}\Lambda^+$ and the bundle Λ^+T^*M is flat. It follows that $b^+=3$ and $\sigma\leqslant 3$.

The equality $b^+=3$ occurs for a *K3 surface*, by definition a simply-connected compact complex surface with $c_1=0$. Any two are known to be diffeomorphic, and an example is a quartic hypersurface K in $\mathbb{CP}^3$. The Chern classes of K are easily computed from the formula $T\mathbb{CP}^3|_K = TK\oplus L^4$; we obtain $c_1=0$

and $c_2 = 6x^2$ where x now denotes the pullback of the generator of $H^2(\mathbb{CP}^3, \mathbb{Z})$. Thus, $\sigma = -16$ and $\chi = 24$, so $b^+ = 3$, $b^- = 19$ and $b_2 = 22$.

Let M be a compact, oriented, simply-connected, 4-manifold, with (as we always assume) a smooth structure. The choice of a Riemannian metric reduces the structure group of the tangent bundle to $SO(4)$, and this lifts to $Spin(4) \cong SU(2) \times SU(2)$ (equivalently, the Stiefel-Whitney class $w_2 \in H^2(M, \mathbb{Z}_2)$ vanishes) if and only if the intersection form (4.11) takes only even values. In this spin case it is known that $\sigma \equiv 0$ mod 16, and in the light of the work of Donaldson and Freedman it is conjectured that

$$\frac{b_2}{|\sigma|} \geqslant \frac{11}{8}$$

[35; 43; 44]. If this is the case, M would necessarily be homeomorphic to a connected sum $mK\#n(S^2 \times S^2)$. By contrast, the topological classes of non-spin 4-manifolds are exhausted by the connected sums (4.14).

The condition (4.15) implies that the manifold M is locally Ricci-flat Kähler. If it is simply-connected then, by Theorem 3.12, there exists an orthonormal basis $\{\omega_1, \omega_2, \omega_3\}$ of parallel sections and M is hyperkähler. In this case, M is necessarily diffeomorphic to T^4 or a K3 surface [58]. Of course, the torus has a flat metric and compatible hyperkähler structure, whereas any K3 surface admits a hyperkähler metric by Yau's deep theorem [110], a new account of which can be found in Joyce's book [63].

The question of exactly which compact smooth 4-manifolds can admit Einstein metrics has been pursued by LeBrun and collaborators, by using Seiberg-Witten theory to refine Corollary 4.12. For example, it can be shown that the topological manifold underlying a K3 surface has infinitely many smooth structures, but the above argument shows that only the standard one admits an Einstein metric. This situation is not untypical [73].

Structures on 4-manifolds

Recall that, at each point, W_+ is a self-adjoint linear transformation of the 3-dimensional space Λ^+. Let its eigenvalues be $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$ with $\sum \lambda_i = 0$, and let $\{\sigma_1, \sigma_2, \sigma_3\}$ be a corresponding basis of orthonormal eigenvectors. The following result can be found in [80; 93]:

Lemma 4.16 *If J is an OCS on M, oriented so that $\omega \in \Gamma(M, \Lambda^+ T^* M)$. Then*

$$\pm\omega = \sqrt{\frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_3}}\sigma_1 \ \pm \ \sqrt{\frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}}\sigma_3. \tag{4.17}$$

Proof We shall show that the fundamental 2-form ω lies in the span of σ_1, σ_3, leaving determination of the coefficients as an exercise. The condition $W'' = 0$

implies that W_+ is represented by the matrix

$$\begin{pmatrix} 2\lambda & x & y \\ x & -\lambda & 0 \\ y & 0 & -\lambda \end{pmatrix}.$$

Computation of its characteristic polynomial reveals that $\lambda = \lambda_2$ is the middle eigenvalue. The corresponding eigenvector is the column vector

$$\sigma_2 = \pm(0, \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}})^T,$$

and the result follows from the fact that ω is represented by $(1,0,0)^T$. □

We shall call the four elements (4.17) the *roots* of the tensor W_+. This terminology is justified by spinor language in which W_+ is represented by a quartic polynomial, and (4.17) are (projectivizations of) the roots of this polynomial. The lemma allows us to perform a classification in terms of the existence of compatible complex structures. There are two cases according as to whether W_+ is identically zero or not, and we proceed to consider each in turn.

Hermitian manifolds with $W_+ \neq 0$

A generic Riemannian metric will not admit any compatible complex structure locally, since the roots of W_+ will determine non-integrable almost-complex structures. Incidentally, this situation shows that the converse of Lemma 4.7(ii) is false, although Hermitian Weyl tensor together with an additional curvature condition is known to imply that $N = 0$ (see Theorem 4.22 below).

A generic Hermitian metric compatible with a given complex structure J will admit only $\pm J$ as a compatible complex structure, as the other root will be non-integrable. A more special situation is that in which two of the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ coincide (as do the roots of W_+ in pairs). This occurs if and only if $W' = 0 = W''$, or equivalently

$$W_+ = \begin{pmatrix} \frac{1}{6}s & 0 & 0 \\ 0 & -\frac{1}{12}s & 0 \\ 0 & 0 & -\frac{1}{12}s \end{pmatrix}. \tag{4.18}$$

We have seen that any Kähler metric has this property, but (4.18) can also hold in other cases.

A Riemannian version of the so-called *Goldberg-Sachs theorem* implies that an Einstein metric satisfies (4.18) (and so is $*$Einstein) if and only if it is Hermitian relative to the eigenform σ_1 [6]. The only compact example known of an Einstein-Hermitian manifold that is not Kähler is $\mathbb{CP}^2$ blown up at one point, with Page's metric that has a non-trivial group of isometries. If there is another, the underlying complex surface must be must be biholomorphic to $\mathbb{CP}^2$ blown up at either 2 or 3 points [72].

Example 4.19 A simple instance of a $*$Einstein non-Einstein metric satisfying (4.18) is the one

$$g = dx^2 + dy^2 + dt^2 + (dz - xdy)^2$$

naturally defined in Example (1.9) [1]. The formal similarity with (2.31) is part of a more general construction [49; 56].

Non-ASD *bihermitian metrics* have the property that the eigenvalues of W_+ are distinct on a dense open set, but that all the almost-complex structures defined by Lemma 4.16 are integrable. In this case, the twistor space of M has four sections that constitute the zero set of the Nijenhuis tensor of the almost-complex structure J_1. Following the initial examples of [67], a general construction of such metrics has been given in [7], and we describe this in the next paragraph. A contrasting situation described in [70] is that in which a 4-manifold has two complex structures determining *opposite* orientations [70].

If $T = \mathbb{R}^4$ and υ is a non-zero element of $\bigwedge^4 T^*$, then the formula $\alpha \wedge \beta = B(\alpha, \beta)\upsilon$ defines a bilinear form B on $\bigwedge^2 T^*$. This bilinear form is known to have signature $(3,3)$ and defines a double covering from $SL(4,\mathbb{R})$ to a connected component of $O(3,3)$ [95]. Moreover, there is a bijective correspondence between 3-dimensional subspaces Λ^+ of $\bigwedge^2 T^*$ on which B is positive-definite, and oriented conformal structures on T. Now let M be an oriented 4-manifold, and suppose that Φ_r, $r = 1, 2, 3$ is a triple of real symplectic forms on M for which $\upsilon = \Phi_r \wedge \Phi_r$ is independent of r, and relative to which the matrix of B is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & p \\ 0 & p & 1 \end{pmatrix}, \quad |p| < 1.$$

It follows that there exists a Riemannian metric g for which $|\Phi_r| = 2$ (in accordance with the convention used in (4.2)), and $\Lambda^+ = \langle \Phi_1, \Phi_2, \Phi_3 \rangle$. Setting $\Lambda^{2,0} = \langle \Phi_0 + i\Phi_r \rangle$ determines an almost-complex structure J_r for $r = 1$ and $r = 2$, with the property that $J_1 J_2 + J_2 J_1 = -2p\mathbb{1}$. However, Lemma 2.29 shows that J_1 and J_2 are both *integrable*, so (M, g, J_1, J_2) is bihermitian.

This method can be used to prove the existence of non-ASD bihermitian structures on any 4-manifold with a hyperhermitian metric. The latter fits into the next category.

Hermitian manifolds with $W_+ = 0$

Recall that M is called *anti-self-dual* (ASD) if W_+ is identically zero. It follows from Lemma 4.16 that this will hold whenever M admits at least three independent OCS's compatible with the orientation, around each point. Conversely, $W_+ \equiv 0$ implies that the twistor space is a complex manifold, and any almost-complex structure on $T_m M$ extends to a complex structure on a neighbourhood of $m \in M$. Thus, there are actually infinitely many OCS's locally. The elementary compact examples are S^4 (with its conformally flat metric) and $\overline{\mathbb{CP}}^2$ (with its symmetric metric of constant holomorphic sectional curvature).

Following results of Poon [90], LeBrun [71], and others, general constructions of ASD metrics were found by Floer [41], Donaldson and Friedman [36], and Taubes [100] who proved

Theorem 4.20 *If M is a compact oriented smooth 4-manifold then $M\#n\overline{\mathbb{CP}}^2$ admits an ASD metric for all sufficiently large n.*

Estimating the minimal n for a given M is non-trivial; for example, $n=14$ suffices for $M=\mathbb{CP}^2$.

One can divide the class of ASD metrics into subcases according to the number of OCS's that exist *globally* on M, which we now suppose to be compact.

The classification of ASD Hermitian surfaces depends crucially on the parity of b_1 [19]. If b_1 is even then the metric is conformally equivalent to a Kähler one, which is therefore SFK. Such metrics were investigated in [65], which established analogues of Theorem 4.20 for SFK under blowing up. If b_1 is odd then M cannot carry a Kähler metric. However, the Lee form is closed and it follows that the metric is *locally* conformally Kähler. The surfaces in question are the so-called Type VII_0 ones.

A classification of bihermitian ASD metrics has been given in [89]; these are metrics with $W_+=0$ but admitting two OCS's J, J' compatible with the orientation, for which the set A of points where $J'=\pm J$ is a proper subset of M. In fact, if $A\neq\emptyset$ then A is a union of complex curves whose existence restricts the possibilities in the type VII_0 case. If $A=\emptyset$ the structure is called *strongly bihermitian* and M is nececessarily hyperhermitian, so that there are then infinitely many OCS's on M. It follows that M is either a Hopf surface or hyperkähler [20]. In the latter case, M is a torus, or a K3 surface with a Calabi-Yau metric.

We remark that in each dimension $4k\geqslant 8$, there exist flat hyperkähler metrics on finite quotients of a torus T^{4k} and at least two compact irreducible hyperkähler manifolds that are not diffeomorphic [16; 86]. Moreover, the Euler characteristic of any compact hyperkähler manifold of dimension $4k$ satisfies $k\chi\equiv 0 \bmod 24$ [94].

Exercises 4.21 (i) Let ∇ denote its Levi-Civita connection of the metric g in Example 4.19, and set $e^i=\sum_j \sigma^i_j\otimes e^j$. Determine the 1-forms σ^i_j, using the formulae $de^i=\sum_j \sigma^i_j\wedge e^j$ and $\sigma^i_j+\sigma^j_i=0$. Compute the curvature of g using the formula $R^i{}_{jk\ell}e^k\wedge e^\ell=d\sigma^i_j-\sum_k \sigma^i_k\wedge\sigma^k_j$ (find $R^i{}_{jk\ell}$ for as many (i,j,k,ℓ) as are necessary to determine R from its known symmetries). Hence, determine the eigenvalues of W_+.

(ii) Complete the proof of Lemma (4.16), and show that σ_2 is proportional to the component of $d\theta$ (the exterior derivative of the Lee form) in $\bigwedge^+$.

Almost-Kähler 4-manifolds

Let (M, g, J) be an almost-Hermitian 4-manifold. In terms of the decomposition of ∇J, the condition that ω be closed is exactly complementary to the integrability of J. The two conditions together imply that $\nabla J = 0$ and the structure is Kähler. We have mentioned above the little that is known about Einstein-Hermitian metrics that are not Kähler, and the situation is analogous for Einstein almost-Kähler (EaK) metrics that are not Kähler. Such metrics admit a compatible non-integrable almost-complex structure J for which $d\omega = 0$, and have been studied in connection with the Goldberg conjecture mentioned in the Introduction.

The following compilation of results is indicative of progress in this area.

Theorem 4.22 *Let (M, g, J) be a compact EaK 4-manifold. Then J is necessarily integrable (and so M is Kähler) if any one of the following conditions applies:*

(i) s is non-negative [98]*;*

(ii) s^* is constant [8]*;*

(iii) $W' = 0$ [9]*;*

(iv) $W'' = 0$ [4].

Turning to the local problem, Example 2.30 provides a simple Einstein strictly almost-Kähler structure. We showed there that the metric $s^3 g$ is hyperkähler, and therefore Einstein. Reverse the orientation of this example by considering the 2-forms ω_i^- formed from (2.32) by changing signs. In particular,

$$\omega_2^- = dx \wedge (du - x\,dv) - s\,ds \wedge dv$$

is a closed 2-form for which the corresponding almost-complex structure I_2^- is *not* integrable. We could have also chosen ω_3^-; the non-integrability follows because $\omega_1^- \neq 0$. Such examples were first discovered in [84], and are completely characterized locally by the condition $W' = 0$ [9]. Different examples appear in [5], though share the property that the metric becomes hyperkähler relative to the opposite orientation.

The metrics described in the previous paragraph are necessarily non-complete. An attempt to construct EaK metrics compatible with the standard symplectic form on $\mathbb{R}^4$ reveals that there is a non-trivial obstruction to extending the 3-jet of such a metric g to a 4-jet. This obstruction derives from the formula

$$\Delta s^* = -\tfrac{1}{4}(3s^* - s)(s^* - s) + 12|W'|^2 - 8|W''|^2 + 4\langle \nabla\Psi, \nabla\omega \rangle, \qquad (4.23)$$

where $\Phi \in [\![\Lambda^{2,0}]\!]$ satisfies $-4W' = \Phi \odot \omega$ that appears in [37]. (Observe that the left-hand side depends on $j_4(g)$ whereas the right-hand side is determined by $j_3(g)$.) Integrating (4.23) leads to (i) above.

When $W'' = 0$, the right-hand side of (4.23) is non-negative, and a maximum principle implies that $s^* - s$ is identically zero. This yields (iv), which remains valid when the Einstein condition $B=0$ is relaxed to the J-invariant Ricci condition $B''=0$, provided $5\chi+6\sigma\neq 0$ [4].

Almost-Kähler manifolds occur naturally in work relating Seiberg-Witten theory to curvature. The main estimate [74] implies that on a compact almost-Hermitian 4-manifold (M,g,J) for which the SW equations have a solution for every metric conformally related to g then

$$\int_M \Big(|W^+| - \tfrac{1}{\sqrt{6}}s\Big)^2 \geqslant 72\pi^2(c^+)^2 \tag{4.24}$$

(see (4.10)). Equality in (4.24) implies that M is almost-Kähler with $W'=0=W''$ (see (4.18)); in this case we have seen that if g is Einstein then it is also Kähler.

Special Kähler manifolds

We use the notation (2.1), and follow closely [42].

Definition 4.25 *A special Kähler manifold is an almost-Hermitian manifold* (M,g,J) *admitting a flat torsion-free connection* ∇ *for which (i)* $\nabla\omega=0$ *and (ii)* $\nabla_1 J=0$.

Given that ∇ is torsion-free, (i) implies that $d\omega=0$ and (ii) that $N=0$. Thus (M,g,J) is indeed Kähler, justifying the terminology.

We shall denore the Levi-Civita connection of g by $\widetilde{\nabla}$ in this final subsection. Whilst $\widetilde{\nabla}$ is also a torsion-free symplectic connection, it is of course not in general flat and so not equal to ∇. For example, we have seen that $\widetilde{\nabla}_1 J=0$ if and only if $\widetilde{\nabla}J=0$ and condition (ii) is only useful for a non-metric connection. The tensor ψ defined in (3.18) is given by

$$\begin{aligned}\psi(X,Y,Z) &= g(J\widetilde{\nabla}_X Y - J\nabla_X Y, Z)\\ &= g(\widetilde{\nabla}_X(JY) - \nabla_X(JY) + (\nabla_X J)Y,\ Z).\end{aligned}$$

It follows that

$$\psi(X,Y,Z) - \psi(X,JY,JZ) = \Phi(X,Y,Z),$$

where Φ is defined in terms of the connection ∇ by (3.21).

Theorem 3.12 implies that there exist 1-forms $\{\alpha^i\}$ such that $\nabla\alpha^i=0$ (so that $d\alpha^i=0$) and $\omega=\sum_{r=1}^n \alpha^r\wedge\alpha^{n+r}$. Since $\tau(\nabla)=0$ there exist charts $(x^1,\dots,y^n)$ such that

$$\omega = \sum_{r=1}^n dx^r\wedge dy^{n+r}, \tag{4.26}$$

and

$$\nabla dx^r = 0 = \nabla dy^r.$$

This leads to an alternative definition of a special Kähler manifold, as an affine manifold with transition functions of the form $(x,y)\longmapsto P(x,y)+(a,b)$ with

$P \in Sp(n,\mathbb{R})$. The associated metric $-w(J\cdot,\cdot)$ need only be pseudo-Riemannian for the theory to work.

If we trivialize the tangent bundle T by the parallel fields $\{\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial y^n}\}$ then the operators in the sequence (3.7) reduce to ordinary d. Thus, locally $J=\nabla\zeta$ where

$$\zeta=\sum_{r=1}^{n}(V^r\frac{\partial}{\partial x^r}+U^r\frac{\partial}{\partial y^r}).$$

Hence

$$J=\sum(dV^r\otimes\frac{\partial}{\partial x^r}+dU^r\otimes\frac{\partial}{\partial y^r}),$$

and $Jdx^r=dV^r$, $Jdy^r=dU^r$. This means that

$$z^r=x^r-iV^r,\quad w^r=y^r-iU^r$$

are holomorphic functions. In this way, one obtains 'conjugate' holomorphic charts $(z^1,\ldots,z^n)$, $(w^1,\ldots,w^n)$.

Define holomorphic functions τ_{rs} by

$$dw^r=\sum_s\tau_{rs}dz^s,\quad\text{or}\quad\tau_{rs}=\frac{\partial w^r}{\partial z^s}.$$

Being a $(1,1)$-form,

$$\omega=\sum_r dx^r\wedge dy^r=J\omega=-\sum_r dU^r\wedge dV^r,$$

whence

$$\sum_r dz^r\wedge dw^r=-d\Big(\sum_r w^r\,dz^r\Big)$$

has zero real part, and so (being of type $(2,0)$) vanishes completely. Then

$$\sum w^r dz^r=d\mathcal{F}$$

for some locally-defined holomorphic function $\mathcal{F}$ that satisfies

$$w^r=\frac{\partial\mathcal{F}}{\partial z^r},\qquad\tau_{rs}=\frac{\partial^2\mathcal{F}}{\partial z^r\partial z^s}=\tau_{sr}.$$

Moreover,

$$2\omega=\Re\mathrm{e}\Big(\sum_r dz^r\wedge d\overline{w}^r\Big)=\Re\mathrm{e}\Big(\sum_{r,s}\overline{\tau}_{rs}dz^r\wedge d\overline{z}^s\Big)=-i\sum_{r,s}(\operatorname{Im}\tau_{rs})dz^r\wedge d\overline{z}^s,$$

so

$$\omega=-\tfrac{1}{2}i\sum_{r,s}\omega_{rs}dz^r\wedge d\overline{z}^s,\qquad\omega_{rs}=\operatorname{Im}\Big(\frac{\partial^2\mathcal{F}}{\partial z^r\partial z^s}\Big).$$

The following is also immediate:

Lemma 4.27 $\omega = -i\partial\overline{\partial}\Big(\operatorname{Im}\sum_r \overline{z}^r \dfrac{\partial\mathcal{F}}{\partial z^r}\Big).$

This means that the *real* function $\operatorname{Im}(\sum \overline{z}^r w^r)$ is a Kähler potential for the metric. By contrast, the function $\mathcal{F}$ is called a holomorphic 'prepotential'. The flat Kähler metric on $\mathbb{C}^n$ is of course special Kähler with $\nabla = \widetilde{\nabla}$, and has $\mathcal{F} = \frac{1}{2}\sum(z^r)^2$.

Let $P\colon T^*_{\mathbb{C}} \to \Lambda^{1,0}$ be the projection (of a 1-form to its component of type $(1,0)$). Thus

$$P = \tfrac{1}{2}(1 - iJ) = \tfrac{1}{2}\sum_r \Big(dz^r \otimes \frac{\partial}{\partial x^r} + dw^r \otimes \frac{\partial}{\partial y^r}\Big).$$

But $P = \sum_r dz^r \otimes \partial_r$ where

$$\partial_r = \frac{\partial}{\partial z^r} = \tfrac{1}{2}\Big(\frac{\partial}{\partial x^r} + i\frac{\partial}{\partial U^r}\Big) = \tfrac{1}{2}\Big(\frac{\partial}{\partial x^r} + \tau_{rs}\frac{\partial}{\partial y^s}\Big)$$

(summation now understood over repeated indices). It follows that

$$\nabla\partial_r = \tfrac{1}{2}\frac{\partial\tau_{rs}}{\partial z^t}dz^t \otimes \frac{\partial}{\partial y^s} \in \Gamma(\Lambda^{1,0}\otimes T_{\mathbb{C}}).$$

Exercises 4.28 (i) Use the last formula to show that $\nabla_{\overline{\partial_r}}\partial_s = 0$ for all r,s. Since $\widetilde{\nabla}J = 0$, the Levi-Civita connection preserves types and $\omega(\widetilde{\nabla}_X\partial_s, \partial_t) = 0$ for all X. Deduce that $\psi(\overline{\partial}_r, \partial_s, \partial_t) = 0$.

(ii) By recalling that ω has the standard form $\sum dx^r \wedge dy^r$, show that

$$\psi(\partial_r, \partial_s, \partial_t) = -\tfrac{1}{4}\frac{\partial\tau_{st}}{\partial z^r}.$$

The exercises tell us that $\psi = \Xi + \overline{\Xi}$, where

$$\Xi = -\tfrac{1}{4}\sum_{r,s,t}\frac{\partial^3\mathcal{F}}{\partial z^r\partial z^s\partial z^t}dz^r \otimes dz^s \otimes dz^t \in \Gamma(S^{3,0})$$

is a *holomorphic* cubic differential. This leads to an expression for the Riemann curvature (i.e. $\widetilde{\nabla}_1 \circ \widetilde{\nabla}$) in terms of $\Xi\otimes\overline{\Xi}$, and the formula for the scalar curvature

$$s = R^i_{jik}g^{jk} = 4|\Xi|^2 \geqslant 0$$

of Lu [77], who deduced that if (M,g) is complete then $\Xi = 0$, so $R = 0$ and g is flat.

Example 4.29 A simple non-trivial special Kähler metric g is the one on the upper half plane $H=\{z:\operatorname{Im}z>0\}$ given by $\mathcal{F}=z^3/6$, so that Ξ is a constant multiple of dz^3. For consistency with above notation, we set $z=x-iV$ and

$$w=y-iU=\mathcal{F}'(z)=\tfrac{1}{2}z^2,$$

so that $\tau=z$. The real part of w is given by

$$y=\tfrac{1}{2}(x^2-V^2), \tag{4.30}$$

and $VdV=xdx-dy$. If we set

$$\phi=\tfrac{1}{3}V^3=\tfrac{1}{3}(x^2-2y)^{3/2} \tag{4.31}$$

then

$$\begin{aligned} g &= V(dx^2+dV^2)\\ &= \frac{1}{V}\Big((x^2+V^2)dx^2-2x\,dx\,dy+dy^2\Big)\\ &= \phi_{xx}dx^2+2\phi_{xy}dxdy+\phi_{yy}dy^2. \end{aligned}$$

Any special Kähler metric can in fact be expressed as

$$g=\sum\frac{\partial^2\phi}{\partial x^i\partial x^j}$$

for a suitable function ϕ of Darboux coordinates [59].

Let M be a special Kähler manifold, and consider its *real* cotangent bundle $\pi\colon T^*M\to M$. Select Darboux coordinates $\{x^1,\dots,x^n,y^1,\dots,y^n\}$ on an open set $\mathcal{U}$ of M, so that (4.26) holds.

We choose to express a point p of $\pi^{-1}\mathcal{U}$ as $\sum(-u^rdx^r+v^rdy^r)$. This gives rise to a tautological 1-form on T^*M:

$$\tau=\sum_{r=1}^n(-u^r\pi^*dx^r+v^r\pi^*dy^r),$$

though we shall omit the pullback symbol π^* by thinking of x^r, y^r as functions on T^*M. The u^r,v^r are 'fibre coordinates', and (for the moment) are unrelated to the upper case the functions we defined on M.

Consider the coordinates (x^r,y^r,u^r,v^r) on $\pi^{-1}\mathcal{U}$, and the 2-forms given by (2.11). In the present context, Ω_2 is none other than $d\tau$ and equals the canonical real symplectic form (as defined on the cotangent bundle of any smooth manifold). Thus, Ω_2 is independent of the choice of coordinates, and extends to a 2-form globally on T^*M.

The 2-form Ω_1 may be written as $\pi^*\omega-\omega^*$, where ω^* is the 'dual' of ω under the identification ω itself provides between T_mM and the tangent space $V=T_p(T_m^*M)\cong T_m^*M$ to the fibre of π at a point p. Moreover, w^* will be independent of the coordinates in the presence of a flat symplectic connection ∇. For in this case, we may write

$$T_p(T^*M) = V \oplus H,$$

where H is the 'horizontal' space determined by ∇ and (by definition) everywhere tangent to sections which are constant linear combinations of dx^r, dy^r.

So far so good, but in order to make the above construction more invariant, we shall replace Ω_3 by

$$\begin{aligned} \Omega_3' = d(J\tau) &= \textstyle\sum(-dU^r \wedge dv^r + dV^r \wedge du^r) \\ &= \operatorname{Im}\textstyle\sum(dw^r \wedge dv^r - dz^r \wedge du^r), \end{aligned}$$

where J is (the pullback of) the complex structure on M. This will ensure that

$$\eta = \Omega_2 + i\Omega_3 = d(\tau + iJ\tau)$$

is (twice) the *holomorphic* symplectic form that exists on T^*M, when the latter is endowed with its natural complex structure I_1 extending J.

The triple $(\Omega_1, \Omega_2, \Omega_3')$ will then define a hyperkähler structure if H is a *complex* subspace of $(T_p(T^*M), I_1)$. This is true if M is special Kähler, since constant linear combinations of dx^r, dy^r (being the real parts of the holomorphic 1-forms dz^r, dw^r) are themselves I_1-holomorphic sections.

We may define a local section

$$s\colon \mathcal{U} \to T^*M$$

by setting $u^r = U^r$ and $v^r = V^r$, so that the notation is amalgamated with what we did earlier. Once we do this,

$$\begin{aligned} s^*\Omega_1 &= \omega - \omega = 0, \\ s^*\Omega_2 &= -\operatorname{Im}(\textstyle\sum dz^r \wedge dw^r) = 0, \\ s^*\Omega_3 &= -2\textstyle\sum dU^r \wedge dV^r = 2\omega. \end{aligned}$$

This shows that s is *bi-Lagrangian* as a submanifold of T^*M. Thus any special Kähler manifold arises locally as a bi-Lagrangian submanifold of $(\mathbb{R}^{4n}, \Omega_1, \Omega_2)$, where $\Omega_1 \Omega_2$ are two standard real symplectic forms. This fact was establised independently by Cortés and Hitchin [15; 59].

Since $0 = s^*\Omega_2 = s^*(d\tau)$, we may (on a possibly smaller open set $\mathcal{U}'$) express $s^*\tau$ as

$$d\phi = \sum_{r=1}^n \left(\frac{\partial\phi}{\partial x^r}dx^r + \frac{\partial\phi}{\partial y^r}dy^r\right)$$

for some real-valued function ϕ so that $u^r = -\partial\phi/\partial x^r$ and $v^r = \partial\phi/\partial y^r$ on M (by analogy to the holomorphic equation $w^r = \partial\mathcal{F}/\partial z^r$). Returning to T^*M, we see that Ω_3' equals

$$\sum_{r=1}^{n}\Big(\frac{\partial^2\phi}{\partial x^r\partial x^s}dx^r\wedge dV^s+\frac{\partial^2\phi}{\partial y^r\partial x^s}dy^r\wedge dV^s+\frac{\partial^2\phi}{\partial x^r\partial y^s}dx^r\wedge dU^s+\frac{\partial^2\phi}{\partial y^r\partial y^s}dy^r\wedge dU^s\Big)$$
$$=\sum_{r=1}^{2n}\frac{\partial^2\phi}{\partial X^r\partial X^s}dX^r\wedge dW^s,$$

in terms of coordinates $X^1,\dots,X^{2n}$ on the base $\mathcal{U}'$ and $W^1,\dots,W^{2n}$ for the fibres. Combined with the more standard expressions for Ω_1 and Ω_2 above this gives an explicit construction of hyperkähler metrics.

Example 4.32 This unites Examples 2.30 and 4.29. The 2-forms (2.32) fall within the above description. To see this, replace s by V, and define y by (4.30). Then

$$\begin{aligned} V\omega_1 &= 2(x^2-y)dx\wedge dv - x(dx\wedge du+dy\wedge dv)+dy\wedge du\\ \omega_2 &= dx\wedge du+dv\wedge dy,\\ -\omega_3 &= dx\wedge dy+du\wedge dv.\end{aligned}$$

If we now take ϕ as in (4.31) then

$$\omega_1=\phi_{xx}dx\wedge dv-\phi_{xy}(dx\wedge du+dy\wedge dv)+\phi_{yy}dy\wedge du.$$

The minus signs can be eliminated by changing the sign of x and v. Observe that the function ϕ appeared as the conformal factor converting a left-invariant hypercomplex metric into a hyperkähler one.

Remark 4.33 (i) Independent proofs [39; 64] exist of the fact that there exists a hyperkähler metric on an open set of the cotangent bundle of *any* real analytic Kähler manifold.

(ii) An analogous theory of 'special complex manifolds' is developed in [3].

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