Self-duality and Exceptional Geometry

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The local isomorphism between the special orthogonal group $SO(4)$ and the product $SO(3)\times SO(3)$ manifests itself in the conformally invariant decomposition of the bundle of 2-forms

$$\Lambda^2 T^*M = \Lambda^2_+ T^*M \oplus \Lambda^2_- T^*M$$

over an oriented Riemannian 4-manifold M. There is a corresponding decomposition of the Weyl curvature tensor $W = W_+ + W_-$, and M is said to be *self-dual* if $W_- = 0$. If M is compact, its signature is given by

$$\tau = \frac{1}{3}p_1 = \frac{1}{12\pi^2}\int_M (|W_+|^2 - |W_-|^2)\upsilon,$$

where υ is the volume form. Consequently, if M is self-dual but not conformally flat, then $\tau > 0$.

Self-duality is the integrability condition for a natural almost complex structure on the 6-dimensional sphere bundle of $\Lambda^2_- T^*M$ [1]. Motivated in part by this result, we study the 7-dimensional total space X of $\Lambda^2_- T^*M$, and characterize curvature conditions on M by means of differential relations between invariant forms on X. First though, we define the exceptional Lie group G_2 using the inclusion $SO(4) \subset G_2$, corresponding to a splitting of dimensions $7 = 3 + 4$. This enables us to construct a family of G_2-structures on X, which amounts to assigning a metric and vector cross product on each tangent space.

There are only two exceptions in the list of holonomy groups of irreducible non-symmetric Riemannian manifolds, namely G_2 and $Spin(7)$ [2,3,5,11]. This explains the importance of G_2-structures, which, in the light of [7], seem to be a little richer than their $Spin(7)$ counterparts. An examination of the structure on X leads us to exhibit there a Riemannian metric with holonomy group G_2, when M is the self-dual Einstein manifold S^4 or $\mathbf{C}P^2$. No such complete metrics were previously known. This, and analogous examples with holonomy G_2 and $Spin(7)$, are the subject of a forthcoming joint paper with R. L. Bryant.

1. Definition of $\mathbf{G}_2$

Let V denote an oriented n-dimensional vector space with a positive definite inner product $<,>$. The inner product extends to one on $\Lambda^k V^*$, and together with the orientation defines a unit volume form $\upsilon \in \Lambda^n V^*$ and an isomorphism $* : \Lambda^k V^* \to \Lambda^{n-k} V^*$, where

$$\sigma(*\tau) = <\sigma, \tau> \upsilon, \qquad \sigma, \tau \in \Lambda^k V^*. \tag{1}$$

Here and in the sequel, an exterior product of differential forms is denoted by their juxtaposition.

Now take $n = 4$ and $k = 2$. Then $*$ is an involution on $\Lambda^2 V^*$, and we consider the 7-dimensional space

$$A = \Lambda^2_- V^* \oplus V^*,$$

where $\Lambda^2_- V^*$ is the -1-eigenspace of $*$. If $\{e^4, e^5, e^6, e^7\}$ is an oriented orthonormal basis of V^*, then $\Lambda^2_- V^*$ is the span of

$$e^1 = e^4 e^5 - e^6 e^7, \quad e^2 = e^4 e^6 - e^7 e^5, \quad e^3 = e^4 e^7 - e^5 e^6. \tag{2}$$

Regarding now $e^1, \ldots, e^7$ as all elements of A, rather than $\Lambda^2 A$, we set

$$\begin{aligned} \varphi' &= e^1 e^2 e^3 \\ \varphi'' &= e^1(e^4 e^5 - e^6 e^7) + e^2(e^4 e^6 - e^7 e^5) + e^3(e^4 e^7 - e^5 e^6). \end{aligned}$$

Then $\varphi = \varphi' + \varphi''$ is the sum of 7 simple 3-forms on a 7-dimensional vector space, and has the following well-known property (see [5]).

Proposition 1 *$G_2 = \{g \in GL(V) : g^*\varphi = \varphi\}$ is a compact Lie group of dimension* 14.

Proof. G_2 is defined above as a closed subgroup of $GL(V)$ containing $SO(4)$. Decreeing $\{e^1, \ldots, e^7\}$ to be an oriented orthonormal basis of A defines an action of $SO(7)$ with Lie algebra

$$\begin{aligned} \mathsf{so}(7) \cong \Lambda^2 A &\cong \Lambda^2(\Lambda^2_- V^*) \oplus (\Lambda^2_- V^* \otimes V^*) \oplus \Lambda^2 V^* \\ &\cong \Lambda^2_- V^* \oplus (V^* \oplus K) \oplus (\Lambda^2_+ V^* \oplus \Lambda^2_- V^*). \end{aligned} \tag{3}$$

Here K denotes the 8-dimensional subspace of $\Lambda^2_- V^* \otimes V^*$ of elements with zero contraction; for example K contains $e^1 \otimes e^4 + e^2 \otimes e^7$ which defines a skew-symmetric endomorphism of V annihilating φ. Hence the Lie algebra $\mathcal{G}_2$ of G_2 contains K, not to mention $\Lambda^2_+ V^*$ and one copy of $\Lambda^2_- V^*$. Now $S^2 A \cong \mathbf{R} \oplus S^2_0 A$, where

$$S^2_0 A \;\cong\; S^2_0(\Lambda^2_- V^*) \oplus \mathbf{R} \oplus V^* \oplus K \oplus S^2_0 V^*$$

is the space of traceless symmetric endomorphisms of A, decomposed into $SO(4)$-modules. Consideration of the action of $K \subset \mathcal{G}_2$ shows that $S^2_0 A$ is G_2-irreducible. Thus

$$\mathcal{G}_2 = \mathsf{so}(4) \oplus K,$$

and it is not hard to check that $G_2 \subset SO(7)$. Q.E.D.

The form φ defines by contraction a two-fold vector cross product

$$m : \Lambda^2 A \longrightarrow A, \qquad (4)$$

of the sort that exists only on a space of dimension 3 or 7 [4]. Using m, $\mathbf{O} = \mathbf{R} \oplus A$ can be identified with the alternative algebra of Cayley numbers, to give the description of G_2 as the group of automorphisms of $\mathbf{O}$. The subspace $\mathbf{H} = \mathbf{R} \oplus \Lambda^2_- V^*$ corresponds to a quaternionic subalgebra, and K may be identified with the tangent space of the quaternionic symmetric space $G_2/SO(4)$, parametrizing all quaternionic subalgebras in $\mathbf{O}$ [9].

Like $S^2_0 A$, the G_2-modules A and $\mathcal{G}_2$ are irreducible, and from (4), the orthogonal complement $\mathcal{G}_2^\perp$ of $\mathcal{G}_2$ in $\mathsf{so}(7)$ must be isomorphic to A. The derivative

$$\delta : \mathrm{End}(A) \cong A \otimes A \hookrightarrow \Lambda^3 A$$

of the action of $GL(V)$ on φ has kernel $\mathcal{G}_2$. It follows that the orbit $GL(V)/G_2$ containing φ is open in $\Lambda^3 A$; in fact there is just one other open orbit, containing the form $\varphi' - \varphi''$, with stabilizer the non-compact form G^* [5]. Anyway, the above remarks establish

Proposition 2 $\quad \Lambda^2 A \cong \mathcal{G}_2 \oplus A, \quad \Lambda^3 A \cong \mathbf{R} \oplus S^2_0 A \oplus A.$

2. Four-dimensional Riemannian Geometry

Let M be an oriented Riemannian 4-manifold. We shall now use the symbols e^4, e^5, e^6, e^7 to denote elements of an oriented orthonormal basis of 1-forms on an open set U of M. Accordingly e^1, e^2, e^3 defined by (2) form a basis of sections over U of $\Lambda^2_- T^*M$. The Levi Civita connection on M induces a covariant derivative ∇ on this vector bundle, and we set

$$\nabla e^i = \Sigma\, \omega^i_j \otimes e^j, \qquad \Omega^i_j = d\omega^i_j - \Sigma\, \omega^i_k \omega^k_j.$$

Summations here and below are exclusively over the range of indices 1,2,3.

Let X denote the total space of $\Lambda^2_- T^*M$; its cotangent space at x admits a splitting

$$T^*_x X = V^o \oplus H^o, \tag{5}$$

where H^o is the annihilator of the horizontal subspaces defined by ∇, and $V^o = \pi^* T^*_m M$, $m = \pi(x)$. A local section $\Sigma\, a^i e^i$ of $\Lambda^2_- T^*M$ is covariant constant iff $\Sigma(da^i + \Sigma\, a^j \omega^j_i) \otimes e^i = 0$, so H^o is spanned by 1-forms

$$f^i = da^i + \Sigma\, a^j \pi^* \omega^j_i,$$

where a^1, a^2, a^3 are now interpreted as fibre coordinate functions on X. Of course V^o is spanned by $\pi^*e^4, \pi^*e^5, \pi^*e^6, \pi^*e^7$.

Omitting the symbol π^*, consider the following invariant forms, defined globally on X, independently of the choice of basis:

$$r = \Sigma\,(a^i)^2$$
$$dr = 2\Sigma\, a^i f^i$$
$$\alpha = \Sigma\, a^i e^i$$
$$d\alpha = \Sigma\, e^i f^i, \quad \beta = f^1 f^2 f^3$$
$$\gamma = e^1 f^2 f^3 + e^2 f^3 f^1 + e^3 f^1 f^2, \quad \upsilon = -\tfrac{1}{6}\Sigma\, e^i e^i$$

For example r is simply the radius squared, α is the tautological 2-form on X, and $\upsilon = e^4 e^5 e^6 e^7$ is the pullback of the volume form on M.

Proposition 3 *(i) M is self-dual if and only if $d\gamma = 2t\upsilon dr$ for (the pullback of) some scalar function t on M; (ii) M is self-dual and Einstein if and only if $d\beta = \frac{1}{2} t\, d\alpha dr$, for some constant t. If t exists in either case, it equals $\frac{1}{12}$ of the scalar curvature of M.*

Proof. We refer the reader to [1] for basic properties of the curvature tensor of a Riemannian 4-manifold. The curvature of the induced connection on the bundle $\Lambda^2_- T^*M$ is determined by the Ricci tensor, and the half W_- of the Weyl tensor which may be regarded as a section of $\Lambda^2_- T^*M \otimes \Lambda^2_- T^*M$. Moreover M is self-dual and Einstein iff

$$\Omega^1_2 = te^3, \quad \Omega^2_3 = te^1, \quad \Omega^3_1 = te^2, \tag{6}$$

where $t = \frac{1}{12}$(scalar curvature). Since the trace-free Ricci tensor essentially belongs to $\Lambda^2_- T^*M \otimes \Lambda^2_+ T^*M$, M is self-dual iff (6) holds modulo elements of $\Lambda^2_+ T^*M$. The proposition is now the result of a computation involving the formulae

$$de^i = \Sigma\, \omega^i_j e^j, \qquad df^i = \Sigma(f^j \omega^j_i + a^j \Omega^j_i).$$

Q.E.D.

Motivated by section 1, we next consider the 3-form

$$\varphi = \lambda^3 \beta + \lambda \mu^2 d\alpha, \tag{7}$$

where λ and μ are scalar functions on X. Observe that

$$\varphi = E^1E^2E^3 + E^1E^4E^5 - E^1E^6E^7 + E^2E^4E^6 - E^2E^7E^5 + E^3E^4E^7 - E^3E^5E^6,$$

where E^i equals λf^i for $i = 1, 2, 3$ and $\mu\, \pi^* e^i$ for $i = 4, 5, 6, 7$, and forms an oriented orthonormal basis of 1-forms for the underlying $SO(7)$-structure on X. In view of (1), we also have

$$\begin{aligned} *\varphi &= E^4E^5E^6E^7 + E^2E^3E^6E^7 - E^2E^3E^4E^5 + E^3E^1E^7E^5 \\ &\qquad - E^3E^1E^4E^6 + E^1E^2E^5E^6 - E^1E^2E^4E^7 \\ &= \mu^4 \upsilon - \lambda^2 \mu^2 \gamma. \end{aligned} \tag{8}$$

Proposition 1 implies

Proposition 4 *If λ and μ are strictly positive everywhere,* (7) *determines a G_2-structure on X, i.e. a G_2-subbundle P of the principal frame bundle of X, whose underlying Riemannian metric has the form $\lambda^2 g^V + \mu^2 g^H$ in terms of the splitting* (5).

3. Torsion considerations

If D denotes the Levi Civita connection of the Riemannian metric in Proposition 4, the quantity $D\varphi$ measures the failure of the holonomy group to reduce to G_2, i.e. the extent to which parallel transport does not preserve the principal subbundle P. Its properties were studied by Fernández and Gray in [7], and we first summarize their approach.

Choose any connection $\tilde{D}$ that reduces to P, so that $\tilde{D}\varphi = 0$. Fix a frame $p \in P$ at the point $x = \pi(p) \in X$, and a vector $v \in T_xX$. The difference $D_v - \tilde{D}_v$ defines, relative to p, an element of the Lie algebra $\mathsf{so}(7)$. The same is true of $D_v\varphi = (D_v - \tilde{D}_v)\varphi$, but since this is independent of the choice of $\tilde{D}$, it actually belongs to the subspace $\mathcal{G}_2^\perp$. Therefore $(D\varphi)_x$ may be regarded as an element of

$$T_x^*X \otimes \mathcal{G}_2^\perp \;\cong\; A \otimes A \;\cong\; \mathbf{R} \oplus \mathcal{G}_2 \oplus S_0^2A \oplus A. \tag{9}$$

Let $W_1X \cong X \times \mathbf{R}$, W_2X, W_3X, $W_4X \cong TX \cong T^*X$ denote the vector bundles associated to P with fibre $\mathbf{R}$, $\mathcal{G}_2$, S_0^2A, A respectively. Corresponding to (9), there is a decomposition

$$D\varphi = w_1 + w_2 + w_3 + w_4,$$

in which w_i is a section of W_iX. Now D is torsion-free, and there exist surjective homomorphisms

$$\begin{aligned} a : T^*X \otimes \Lambda^3T^*X &\longrightarrow \Lambda^4T^*X \cong W_1X \oplus W_3X \oplus W_4X \\ a^* : T^*X \otimes \Lambda^3T^*X &\longrightarrow \Lambda^5T^*X \cong W_2X \oplus W_4X, \end{aligned}$$

such that $d\varphi = a(D\varphi)$ and $d{*}\varphi = a^*(D\varphi)$ (cf. Proposition 2). Thus

Proposition 5 [7] *With the above identifications, $d\varphi = (w_1, w_3, w_4)$, and $d{*}\varphi = (w_2, w_4)$, so $D\varphi = 0$ if and only if $d\varphi = 0 = d{*}\varphi$.*

Call a differential form on X of type (p, q) if, at each point, it is built up from forms on the base of degree p and forms of degree q involving f^i. Endow X with the G_2-structure of Proposition 4, with λ and μ arbitrary positive scalar functions on X. Then $d\varphi$, unlike $*\varphi$, has no component of type $(4, 0)$. Moreover $\varphi d\varphi = 0$, whence $d\varphi$ has no component in the subbundle

$W_1X \subset \Lambda^4T^*X$, and we always have $w_1 = 0$. Further components of $D\varphi$ can be eliminated by a suitable choice of λ and μ.

Theorem *(i) If M is self-dual, an open set of X admits a G_2-structure with $D\varphi = w_3$; (ii) if M is self-dual and Einstein, an open set of X admits a G_2-structure with $D\varphi = 0$.*

Proof. We apply Proposition 3. If M is self-dual, we seek λ, μ such that

$$d{*}\varphi = d(\mu^4)\upsilon - d(\lambda^2\mu^2)\gamma - \lambda^2\mu^2 2t\upsilon dr,$$

vanishes. Taking $\lambda\mu = c =$ constant, we obtain a solution

$$\mu = (2c^2tr + d)^{\frac{1}{4}}, \quad \lambda = c(2c^2tr + d)^{-\frac{1}{4}}, \tag{10}$$

where d is another constant. If M is also Einstein, then $dt = 0$ and

$$d\varphi = d(\lambda^3)\beta + \lambda^3\tfrac{1}{2}td\alpha dr + d(\lambda\mu^2)d\alpha = 0.$$

Note that λ, μ can only be strictly positive on all of X if t is everywhere non-negative. Q.E.D.

In [7] it is shown that any minimally embedded hypersurface of $\mathbf{R^8}$ also has a G_2-structure with $D\varphi = w_3$. A contrasting example with $D\varphi = w_2 \neq 0$ has been found in [6]. We remark that in general w_2 is the obstruction to the existence of a short elliptic complex

$$0 \to C^\infty(X) \xrightarrow{\text{grad}} C^\infty(X, TX) \xrightarrow{\text{curl}} C^\infty(X, TX) \xrightarrow{\text{div}} C^\infty(X) \to 0,$$

on X whose operators are manufactured using D and (4) in analogy with the 3-dimensional case. Indeed, if $f \in C^\infty(X)$ is a function, and $v \in C^\infty(X, TX)$ is a vector field, $\operatorname{curl}(\operatorname{grad} f) = m(D \wedge (\operatorname{grad} f))$ vanishes identically, but $\operatorname{div}(\operatorname{curl} v)$ equals the contraction of Dv with w_2. We conjecture that a complex of this sort can be defined on X, using only the self-dual conformal structure of M. Topological consequences of the existence of a self-dual metric with t non-negative have been given by LeBrun [10].

Self-dual Einstein metrics have been generated by quaternionic Kähler reduction [8]. However a theorem of Hitchin states that a complete Riemannian 4-manifold which is self-dual, Einstein and of positive scalar curvature is necessarily isometric to the sphere S^4, or the complex projective plane $\mathbf{C}P^2$ [3, 13.30]. In either of these two cases, the Riemannian metric

$$(2tr+1)^{-\frac{1}{2}}g^V + (2tr+1)^{\frac{1}{2}}g^H$$

on X corresponding to the solution (10) with $c = d = 1$ is complete, essentially because $\int_0^\infty (2tr+1)^{-\frac{1}{4}} d(r^{\frac{1}{2}})$ diverges. Because $D\varphi = 0$, the holonomy group H is contained in G_2, which in turn implies that the Ricci tensor is zero [3]. Furthermore, the respective groups $SO(5), SU(3)$ act as isometries on X with generic orbits of codimension 1. Consideration of the induced action on a hypothetical space of covariant constant 1-forms shows that X is locally irreducible, and it follows that $H = G_2$ [5]. In conclusion:

Corollary *The total space of $\Lambda^2_- T^* S^4$ and $\Lambda^2_- T^* \mathbf{C}P^2$ admits a complete Ricci-flat Riemannian metric with holonomy equal to G_2.*

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