Solving a general system 29/03

SUMMARY

Consider a linear system in matrix form

$$AX = B\,, \qquad A \in \mathbb{R}^{m,n}, \quad X \in \mathbb{R}^{n,1}, \quad B \in \mathbb{R}^{m,1}.$$

The study of an example with $m = n = 3$ led us to formulate the so-called theorems of Rouché and Capelli:

(RC1) The solution has at least one solution if and only if $\mathrm{rank}(A\,|\,B) = \mathrm{rank}\,A$**. This is obviously true if $B = \mathbf{0}$.**

(RC2) If $r = \mathrm{rank}\,A = \mathrm{rank}(A\,|\,B)$ then the general solution depends on $n - r$ free parameters. We can add to a particular solution any solution of the homogeneous equation $AX = \mathbf{0}$.

Explanation. If $C_1, \dots, C_m$ are the columns of A, the system can be written as an equation between column vectors:

$$x_1 C_1 + \cdots + x_n C_n = B.$$

This has a solution if and only if $B \in \mathscr{L}\{C_1, \dots, C_n\}$, so the rank of A must not increase if we add B as a final column.

To find the general solution, it is best to apply ERO's to $(A\,|\,B)$ to make it super-reduced, with 0's above and below the r markers. Any zero row for A must have a 0 on the right too.

We can make the $n - r$ 'unmarked' variables free parameters, and then express the 'marked' variables in terms of them.