Gauss-Jordan method 24/03

SUMMARY

To solve a linear system $AX = B$, we apply elementary row operations to the matrix

$$M = (\ A \mid B) \in \mathbb{R}^{m,n+1}$$

(or to A if $B = \mathbf{0}$). These ERO's do not change the solutions of the system.

The aim is to obtain a matrix M' that is
step-reduced, with markers satisfying (M1), (M2), (M3),
or (better!) a matrix M'' that is
super-reduced, satisfying (M2), (M3), (M4), (M5)$^{\Rightarrow \text{(M1)}}$.
The linear systems represented by M, M' and M'' all have the same set of solutions.

The _rank_ of M is the number of non-zero rows (or markers) of M' or M'', and does not depend on the method of reduction. It is known that the super-reduced (or RREF) form M'' is unique.

Example. The following matrices all have rank $r = 2$:

$$M = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 3 & 5 & 8 & 0 \\ 13 & 21 & 34 & 0 \end{array}\right) \qquad \left\{\begin{array}{rcl} x + y + 2z &=& 0 \\ 3x + 5y + 8z &=& 0 \\ 13x + 21y + 34z &=& 0 \end{array}\right.$$

$$M' = \left(\begin{array}{ccc|c} \boxed{1} & 1 & 2 & 0 \\ 0 & \boxed{2} & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left\{\begin{array}{rcl} x + y + 2z &=& 0 \\ 2y + 2z &=& 0 \end{array}\right.$$

$$M'' = \left(\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left\{\begin{array}{rcl} x \quad + z &=& 0 \\ y + z &=& 0 \end{array}\right.$$

The rank equals the number of independent rows in each matrix and the number of effective equations in each system. Here, the homogeneous system has $\infty^{n-r} = \infty^1$ solutions:

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t \\ -t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$