Elementary row operations 22/03

SUMMARY

The following system of 3 equations and 3 unknowns:

$$\begin{cases} x + 2y = 1 \\ -x + 2y + 2z = 0 \\ x - y - z = 3 \end{cases}$$

is equivalent to the matrix equation $AX = B$ in which $\det A = 2$. The system therefore has a unique solution $X = A^{-1}B$. But to find this, we apply ERO's to the matrix

$$(A \mid B) = \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ -1 & 2 & 2 & 0 \\ 1 & -1 & -1 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 4 & 2 & 1 \\ 1 & -1 & -1 & 3 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 4 & 2 & 1 \\ 0 & -3 & -1 & 2 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{4} \\ 0 & -3 & -1 & 2 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{11}{4} \end{array}\right)$$

The last matrix confirms that there is a unique solution. To find it, we can continue to simplify the left part of the matrix, working backwards from bottom right:

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{11}{2} \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{11}{2} \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{11}{2} \end{array}\right)$$

The solution is therefore

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -\frac{5}{2} \\ \frac{11}{2} \end{pmatrix}.$$

An ERO of type (iii) (swapping two rows) is needed when, for example, the first row of A starts with a 0.