Tangents and normals 23/06

Let $f(x,y) = x^3 - 3xy^2 = \mathrm{Re}[(x+iy)^3]$.

$(0,0)$ is a critical point, a 'monkey saddle' with $\widehat{Hf} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

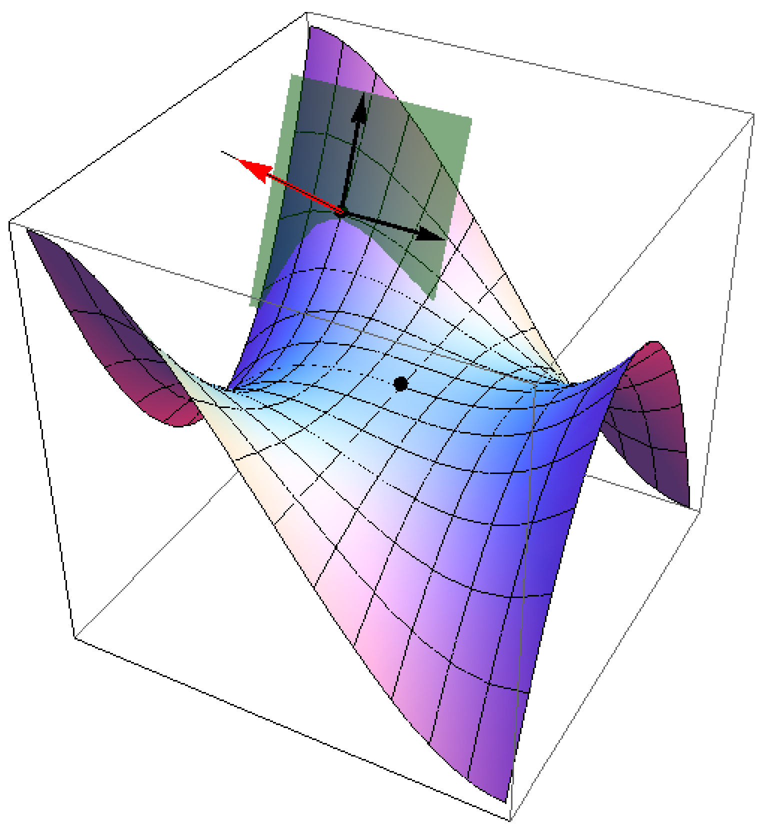
The graph of f is the surface $z - f(x,y) = 0$ and contains $P = (-1,1,2)$. Two tangent vectors at P are

$$\boldsymbol{\sigma}_u = (1, 0, \widehat{f_x}) = (1,0,0), \qquad \boldsymbol{\sigma}_v = (0,1,\widehat{f_y}) = (0,1,6);$$

they point along the graph sliced by the respective vertical planes $y=1$, $x=-1$ through P. A normal vector at P is

$$\mathbf{n} = \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (-\widehat{f_x}, -\widehat{f_y}, 1) = (0,-6,1);$$

this is also the gradient of $z - f(x,y)$ at P.



The equation of the tangent plane at P is

$$z = 2 + 6(y-1) \quad \textbf{or} \quad 6y - z = 4.$$

The parametric equation of the normal line at P is

$$(x,y,z) = (-1,1,2) + t(0,-6,1) = (-1,\ 1-6t,\ 2+t).$$