Parametrization of surfaces 20/06

Just as a mapping $\boldsymbol{\gamma}\colon [a,b] \to \mathbb{R}^3$ parametrizes a curve in space, so a mapping

$$\boldsymbol{\sigma}\colon \mathbb{R}^2 \supseteq D \longrightarrow \mathbb{R}^3$$
$$\boldsymbol{\sigma}(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$$

parametrizes a surface S. We assume that each component function is differentiable at each point of the domain D.

If we fix v or u then $\boldsymbol{\sigma}$ defines a system of curves lying on S, and u, v become *curvilinear coordinates* on the surface. For example,

$$\boldsymbol{\sigma}(u,v) = (\cos u \cos v,\ \sin u \cos v,\ \sin v) = (x,y,z)$$

belongs to the sphere S: $x^2+y^2+z^2 = 1$. The formula for $\boldsymbol{\sigma}$ can be deduced by rotating the circle $(\cos v,\ 0,\ \sin v)$ in the xz plane by an angle u around the z-axis. Hence, u, v represent longitude and latitude, and one takes $0 \leqslant u < 2\pi$, $-\frac{\pi}{2} \leqslant v \leqslant \frac{\pi}{2}$ to cover every point of S :

$u \mapsto \boldsymbol{\sigma}(u, v_0)$ is a parallel (a circle of constant latitude v_0)
$v \mapsto \boldsymbol{\sigma}(u_0, v)$ is a meridian ($\frac{1}{2}$ circle of const longitude u_0.)

In general, the partial derivatives

$$\boldsymbol{\sigma}_u = \frac{\partial\sigma}{\partial u} = (x_u, y_u, z_u)$$
$$\boldsymbol{\sigma}_v = \frac{\partial\sigma}{\partial v} = (x_v, y_v, z_v)$$

represent tangent vectors to the curves $v=$const, $u=$const. We say that $\boldsymbol{\sigma}$ is *regular* at (u_0, v_0) if

$$\mathbf{n} = \widehat{\boldsymbol{\sigma}_u} \times \widehat{\boldsymbol{\sigma}_v} = \boldsymbol{\sigma}_u(u_0, v_0) \times \boldsymbol{\sigma}_v(u_0, v_0)$$

is non-zero. It is then a normal vector to S at $P_0 = \boldsymbol{\sigma}(u_0, v_0)$. In the case of the sphere, $\boldsymbol{\sigma}$ is regular if $|v| < \pi/2$, but at the north and south poles we have $\|\boldsymbol{\sigma}_u\| = 0$. A calculation showed that $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (\cos v)\,\boldsymbol{\sigma}(u,v)$.