Examples and interpretations 16/06

1. The equation of the sphere S with centre $C = (1,2,3)$ is

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = r^2$$

If S passes through $P = (4,2,7)$ then $3^2 + 4^2 = r^2$ so $r = 5$ and

$$x^2 + y^2 + z^2 - 2x - 4y - 6z - 11 = 0.$$

Note that (i) $(\nabla f)(P)$ is a multiple of $\vec{CP}$, and the tangent plane at P is $3(x-1) + 6(z-7) = 0$; (ii) a quadric whose equation has one or more mixed terms yz, zx, xy can *never* be a sphere.

2. Taylor expansion of $f(x,y) = 3x + 4ye^x - 5\log y$ at $(0,1)$ gives

$$\begin{aligned} f(x,y) &= 4 + 7x - (y-1) + \underbrace{\tfrac{1}{2}(4x^2 + 8x(y-1) + 5(y-1)^2) + R} \\ &= 5 + 7x - y \qquad + \qquad g(x,y) \end{aligned}$$

where $g(x,y)$ is the height of the graph above the tangent plane $z = 5 + 7x - y$. In contrast to f, the function g does have a critical point at $(0,1)$ and its behaviour is governed by the Hessian of f, giving (in this case) a local minimum. So the graph of f lies entirely *above* its tangent plane near to $P = (0,1,4)$, touching it only at P.

3. To find the extreme values of a function $f(x,y)$ on a region of the plane with boundary, one has to compute the values of f at local maxima and minima within the region, and compare these to the values of f on the boundary.

Consider $f(x,y) = x^2 + (y-1)^2$ on the region $0 \leqslant y \leqslant 4 - x^2$. There is one critical point, namely at $(0,1)$ where f achieves an absolute minimum of 0. On the parabola $y = 4 - x^2$, f becomes $g(x) = x^2 + (3-x^2)^2 = x^4 - 5x^2 + 9$, and $g'(x) = 0$ if $x = 0$ or $x = \pm\sqrt{2/5}$, with respective values $g(x) = 9$ and $g(x) < 9$. On the axis $y = 0$ we get maximum values of $1 + (\pm 2)^2 = 5$. So the absolute maximum of f occurs at $(0,4)$ where $f = 9$.