Quadrics, conics and lines 14/06

An arbitrary conic $ax^2+2bxy+cy^2+2dx+2ey+f=0$ is the intersection of the plane $z=1$ with the quadric

$$ax^2+cy^2+fz^2+2eyz+2dzx+2bxy=0,$$

equivalently

$$(x\ y\ z)\begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0.$$

We can diagonalize the 3×3 matrix by a rotation, and the equation becomes

$$\lambda_1 X^2+\lambda_2 Y^2+\lambda_3 Z^2=0.$$

If the eigenvalues λ_i are all non-zero, we may assume that $\lambda_1,\lambda_2>0$ and $\lambda_3<0$, so that the quadric is an *elliptical cone*. It is easy to visualize plane sections of such a cone that are hyperbolas, ellipses, parabolas or two intersecting lines.

Proposition. Let α,ℓ be skew lines. If ℓ is rotated around α (as axis) then ℓ sweeps out a hyperboloid (of one sheet).

Example. Take α the z-axis and ℓ (in parametric form) to be

$$(x,y,z)=(\underbrace{1,\ t},\ t).$$

Rotate ℓ by applying the matrix $\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$ to obtain

$$\ell_\theta=\{(\cos\theta-t\sin\theta,\ \sin\theta+t\cos\theta,\ t):t\in\mathbb{R}\},$$

a line which (for each fixed θ) lies on the hyperboloid

$$x^2+y^2-z^2=1,$$

as does the line m_θ with $z=-t$ in place of $z=t$.

A cone is formed from straight lines which pass through its vertex and some fixed conic. For example,

$$\{(s,\ st^2,\ st) : s, t \in \mathbb{R}\}$$

joins the origin to a parabola in the plane $x = 1$. **It consists of all the points of the quadric**

$$z^2 = xy \quad \textbf{or} \quad X^2 - Y^2 - 2z^2 = 0$$

(where $x = (X-Y)/\sqrt{2}$ **and** $y = (X+Y)/\sqrt{2}$ **defines a rotation) minus the points** $(0, u, 0)$ **for** $u \neq 0$.

The image shows the two families of straight lines present in the saddle-shaped hyperbolic paraboloid $z = xy$. **Each green line (drawn in the lecture and including the** y**-axis) is**

$$\{(s,\ t,\ st) : t \in \mathbb{R}\}, \quad s \ \textbf{constant,}$$

and each red line (including the x**-axis) is**

$$\{(s,\ t,\ st) : s \in \mathbb{R}\}, \quad t \ \textbf{constant.}$$