Critical points 09/06

Let $P_0 = (x_0, y_0)$ and $h = x - x_0,\ k = y - y_0$.

Proposition. If the second partial derivatives are continuous in a region containing P_0 then $f(x,y)$ equals

$$f(x,y) = \boxed{\widehat{f} + h\widehat{f_x} + k\widehat{f_y} + \tfrac{1}{2}\left(h^2\widehat{f_{xx}} + 2hk\widehat{f_{xy}} + k^2\widehat{f_{yy}}\right)} + R$$

such that $R/(h^2+k^2) \to 0$ as $(h,k) \to (0,0)$.

Definition. P_0 is a ***critical*** or ***stationary*** point if $\widehat{\nabla f} = 0$, or equivalently $\widehat{f_x} = 0 = \widehat{f_y}$.

In this case, the Taylor expansion to second order becomes

$$\boxed{\widehat{f} + \tfrac{1}{2}\left(h^2\widehat{f_{xx}} + 2hk\widehat{f_{xy}} + k^2\widehat{f_{yy}}\right)}$$
$$= f(x_0,y_0) + \tfrac{1}{2}(h\ \ k)\begin{pmatrix} \widehat{f_{xx}} & \widehat{f_{xy}} \\ \widehat{f_{xy}} & \widehat{f_{yy}} \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix},$$

and the behaviour of f near P_0 is governed by the Hessian matrix $A = \widehat{Hf}$ and its associated quadratic form.

Let λ_1, λ_2 be the eigenvalues of A.

- If $\lambda_1, \lambda_2 > 0$ then f has a ***local minimum*** at P_0, meaning
  $f(x_0,y_0) \leqslant f(x,y)$ for all (x,y) close to P_0.
- If $\lambda_1, \lambda_2 < 0$ then f has a ***local maximum*** at P_0.
- If $\lambda_1\lambda_2 < 0$ then f has a ***saddle point*** at P_0.

In the first two cases, $\det A > 0$. If $\det A = 0$, one can say nothing without looking to third order. A local minimum or maximum is called an ***extremum***, and any such point must be critical.

Example. We have seen that $(0,0)$ is an ***absolute maximum*** for $f(x,y) = \cos(x+2y)$, though $\widehat{Hf}$ has eigenvalues $0, -5$.

Explanations. **Let** $A = \widehat{Hf} = (Hf)(x_0, y_0)$. **We can diagonalize** A **by a rotation:**

$$P^{-1}AP = D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad P^{-1} = P^T, \quad \det P = 1.$$

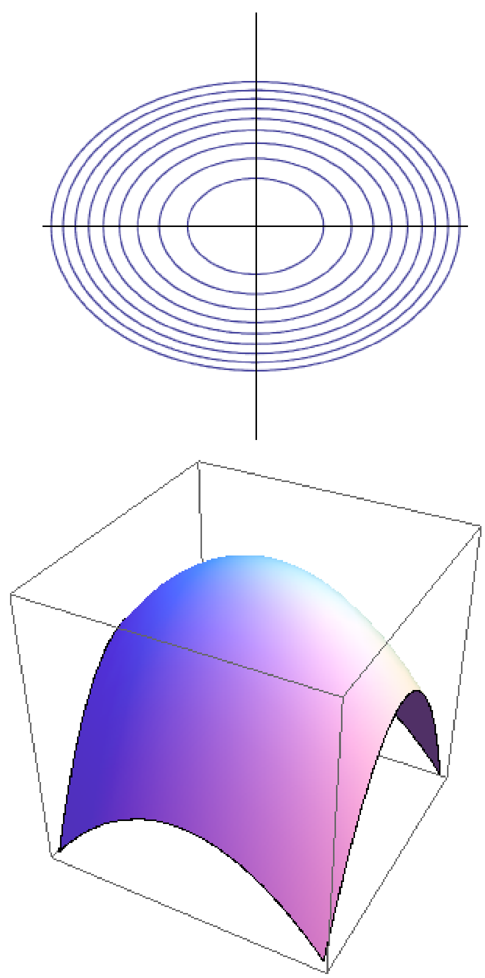
We have new coordinates ℓ, m, **with** $h^2 + k^2 = \ell^2 + m^2$ **and**

$$f(x,y) = \boxed{f(x_0,y_0) + \tfrac{1}{2}\left(\lambda_1 \ell^2 + \lambda_2 m^2\right)} + R,$$

but we can ignore R **if both eigenvalues are non-zero.**

The level curves of the Taylor polynomials in ℓ, m **are *conics*, and the graphs of these polynomials are *quadrics*:**

$\lambda_1 = -1,\ \lambda_2 = -2$



$\lambda_1 = -1,\ \lambda_2 = 2$