First Taylor expansions 07/06

Given a function $f\colon \mathbb{R}^2 \supset D \to \mathbb{R}$, FIX a point

$$P_0 = (x_0, y_0) \in D.$$

I shall use the following notation:

$$\begin{aligned} \widehat{f} &= f(x_0, y_0) = \textbf{value of } f \textbf{ at } P_0 \\ \widehat{f_x} &= f_x(x_0, y_0) = \textbf{slope of } x \mapsto f(x, y_0) \textbf{ at } x_0 \\ \widehat{f_y} &= f_y(x_0, y_0) = \textbf{slope of } y \mapsto f(x_0, y) \textbf{ at } y_0 \end{aligned}$$

These are all *numbers*. The slopes are those of curves, slices of the graph of f that lies above D in space.

The tangent plane to the graph at P_0 has equation

$$\begin{aligned} z &= z_0 + a(x - x_0) + b(y - y_0) \\ &= \widehat{f} + \widehat{f_x}(x - x_0) + \widehat{f_y}(y - y_0) \\ &= \boxed{\widehat{f} + h\widehat{f_x} + k\widehat{f_y}} \end{aligned}$$

where $h = x - x_0$ and $k = y - y_0$. The displayed expression is the *Taylor expansion of f to first order at* P_0.

Proposition. If f_x, f_y exist and are continuous in a region containing P_0 then

$$f(x, y) = \widehat{f} + h\widehat{f_x} + k\widehat{f_y} + R,$$

where the *remainder* $R = R(x, y)$ satisfies

$$R \Big/ \sqrt{h^2 + k^2} \to 0 \quad \textbf{as} \quad (h, k) \to (0, 0).$$

Equivalently,

$$\frac{\left| f(x, y) - f(x_0, y_0) - \widehat{\nabla f} \cdot (h, k) \right|}{\|(h, k)\|} \to 0 \quad \textbf{as} \quad (h, k) \to \mathbf{0}.$$

We say that f is *differentiable* at P_0, and its derivative is represented by the vector $\widehat{\nabla f} = (f_x(x_0, y_0),\ f_y(x_0, y_0))$.

Example. The graph of $f(x,y) = \cos(x+2y)$ is shown in red. We shall find Taylor expansions at the origin $P_0 = (0,0)$, so here $h = x$ and $k = y$.

To first order, $f(x,y)$ equals

$$\widehat{f} + h\widehat{f_x} + k\widehat{f_y} = f(0,0) + 0 + 0 = 1.$$

The tangent plane (shown in blue) is horizontal, confirming that P_0 is a critical point: $(\nabla f)(0,0) = 0$.

To second order, $f(x,y)$ equals

$$\begin{aligned}\widehat{f} + h\widehat{f_x} + k\widehat{f_y} + \tfrac{1}{2}(h^2\widehat{f_{xx}} + 2hk\widehat{f_{xy}} + k^2\widehat{f_{yy}}) &= 1 - \tfrac{1}{2}(x^2 + 4xy + 4y^2) \\ &= 1 - \tfrac{1}{2}(x+2y)^2.\end{aligned}$$

The graph of this quadratic polynomial is the green surface.