Second order partial derivatives 06/06

We have seen that the first partial derivatives of a function $f\colon \mathbb{R}^2 \supseteq D \to \mathbb{R}$ are used to define the tangent plane to its graph at a given point P_0. This gives a first approximation to f at P_0.

One can define various higher order partial derivatives, that can be used in a similar way. We shall concentrate on

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial x}\Big) = (f_x)_x = f_{xx} \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial y}\Big) = (f_y)_x = f_{yx} \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial x}\Big) = (f_x)_y = f_{xy} \\
\frac{\partial^2 f}{\partial y^2} &= f_{yy}.
\end{aligned}$$

These are all functions, whose values at a given point are defined by limits such as

$$f_{xx}(x_0, y_0) = \lim_{h \to 0} \frac{f_x(x_0 + h, y_0) - f_x(x_0, y_0)}{h}.$$

But it is not normally necessary to revert to this definition.

Example. If $r = \sqrt{x^2 + y^2}$ then

$$f(x, y) = \log r = \tfrac{1}{2}\log(x^2 + y^2),$$

and

$$f_x = \frac{r_x}{r} = \frac{x}{r^2}, \qquad f_y = \frac{y}{r^2}.$$

It was shown that f satisfies Laplace's equation

$$f_{xx} + f_{yy} = 0.$$

If $f(x,y) = p(x)q(y)$ **then**

$$f_{xy} = p'(x)q'(y) = f_{yx}.$$

For functions one normally encounters (at least, those with continuous second derivatives), it is always true that

$$f_{xy} = f_{yx}.$$

It follows that

$$(Hf)(x_0,y_0) = \begin{pmatrix} f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0) \\ f_{yx}(x_0,y_0) & y_{yy}(x_0,y_0) \end{pmatrix} \in \mathbb{R}^{2,2}$$

is a *symmetric* matrix. It is called the *Hessian* of f **at** P_0.

A more complicated example. Let $D = \{(x,t) : t > 0\}$, **and**

$$f(x,t) = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}}.$$

Then

$$f_t = (-\tfrac{1}{2}t^{-3/2} + \tfrac{1}{4}x^2 t^{-5/2})e^{-\frac{x^2}{4t}},$$

and it was shown that f **satisfies the equation**

$$f_t = f_{xx}$$

(similar to that representing the temperature f **at time** $t > 0$ **and position** x **along a heated rod). Now fix** $P_0 = (0,1)$. **Since**

$$f_{xx}(0,1) = -\tfrac{1}{2}, \quad f_{xt}(0,1) = 0, \quad f_{tt}(0,1) = \tfrac{3}{4},$$

the Hessian of f **at** P_0 **equals**

$$\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}.$$