Consider $F: \mathbb{R}^3 \supseteq D \to \mathbb{R}$**. Suppose the partial derivatives**

$$F_x = \frac{\partial F}{\partial x}, \quad F_y = \frac{\partial F}{\partial y}, \quad F_z = \frac{\partial F}{\partial z}$$

exist, and that

$$\nabla F = (F_x, F_y, F_z)$$

is non-zero at all points of D**.**

Theorem. Fix $c \in f(D)$**. Then**

$$S = F^{-1}(c) = \{(x, y, z) \in \mathbb{R}^3 : F(x, y, z) = c\}$$

is a 'smooth' surface. Moreover if $P = (x_0, y_0, z_0) \in S$ **then the vector** $(\nabla F)(x_0, y_0, z_0)$ **is *orthogonal* to** S **at** P**.**

Examples. If $F(x, y, z) = x^2 + 4y^2 + 16z^2$ **and** $c = 16$ **then** S **is an ellipsoid. If** $F(x, y, z) = x^3 - y^2$ **and** $c = 0$ **then** S **has contains the** z**-axis (**$x = 0 = y$**) as a sharp edge.**

Suppose that $\gamma(t)$ **is a straight line such that**

$$P_0 = \gamma(0) = (x_0, y_0, z_0) \in S,$$

and consider the values

$$F(t) = (F \circ \gamma)(t) = F(x_0 + tA,\ y_0 + tB,\ z_0 + tC)$$

of F **along this line. The line is *tangent* to** S **if and only if** $F'(0) = (\nabla F) \cdot (A, B, C)$ **is zero. These tangent lines generate the *tangent plane***

$$a(x - x_0) + b(y - y_0) + c(z - x_0) = 0$$

to S **at** P_0 **whose normal vector**

$$\mathbf{n} = (a, b, c) = (F_x(P_0),\ F_y(P_0),\ F_z(P_0))$$

can therefore be taken to be the gradient computed at P_0**.**

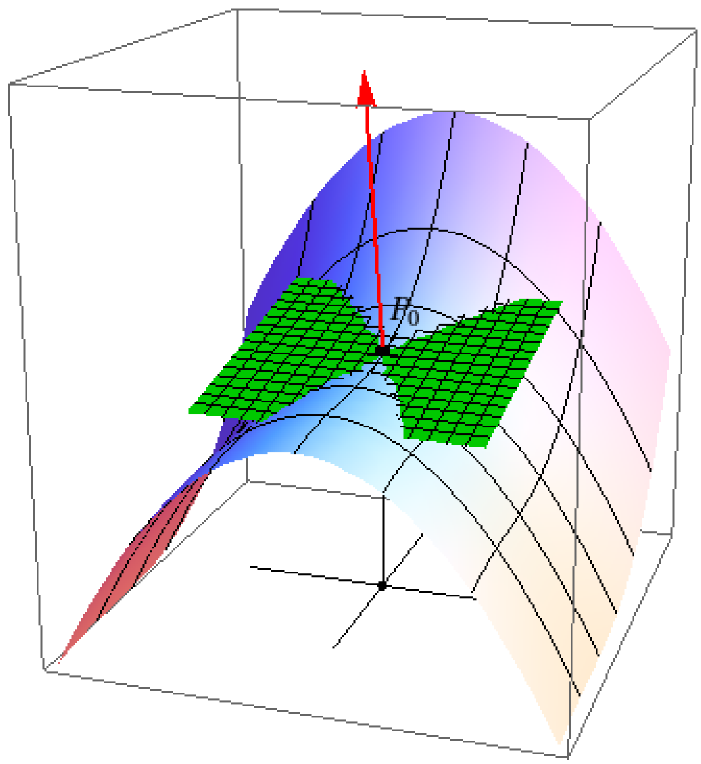
Suppose that $f\colon \mathbb{R}^2 \supseteq D \to \mathbb{R}$. The ***graph*** of f is the surface $z = f(x, y)$. If we set

$$F(x, y, z) = f(x, y) - z,$$

it is the 'level surface' $F(x, y, z) = 0$, and

$$-\nabla F = (-f_x, -f_y, 1)$$

points upwards, everywhere orthogonal to the graph. (In the picture, the red arrow represents $-\nabla F$ computed at a point $P_0 = (x_0, y_0, z_0)$ on the graph.)



Fix (x_0, y_0) and set $a = f_x(x_0, y_0)$ and $b = f_y(x_0, y_0)$. The ***tangent plane*** to the graph at (x_0, y_0) has equation

$$a(x - x_0) + b(y - y_0) - (z - z_0) = 0,$$

where $z_0 = f(x_0, y_0)$. This plane (in green) approximates the graph of f at P_0 and is itself the graph

$$z = z_0 + a(x - x_0) + b(y - y_0)$$

of a simpler function (a linear mapping plus a constant).