Functions to and from $\mathbb{R}^n$ 24/05

Let $F\colon D \to \mathbb{R}^m$ be a function (mapping) where $D \subseteq \mathbb{R}^n$.

Examples:

$\gamma\colon\ [0,2\pi] \to \mathbb{R}^3, \qquad t \mapsto (\cos t,\ \sin t,\ t)$

$f\colon\ \mathbb{R}^3 \setminus \{\mathbf{0}\} \to \mathbb{R}, \qquad (x,y,z) \mapsto \frac{1}{x^2+y^2+z^2} = \frac{1}{r^2}$

$f_A\colon\ \mathbb{R}^n \to \mathbb{R}^m, \qquad X \mapsto AX, \quad A \in \mathbb{R}^{m,n}$ (a linear map)

Notation. We shall write $\mathbf{v} = (x,y,z) \in \mathbb{R}^3$
but $\mathbf{x} = (x_1,\dots,x_n)$, $\mathbf{y} = (y_1,\dots,y_n) \in \mathbb{R}^n$

Definition. The *distance* between $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ is

$$\|\mathbf{x}-\mathbf{y}\| = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}.$$

This is the usual distance for $n = 2,3$ by Pythagoras:

$$\operatorname{dist}\big((x,y),(x',y')\big)^2 = (x-x')^2 + (y-y')^2,$$

and generalizes the absolute value for $n = 1$:

$$\operatorname{dist}(5,-2) = |5-(-2)| = 7.$$

Let $F\colon \mathbb{R}^n \supseteq D \to \mathbb{R}^m$, so

$$\begin{aligned} F(\mathbf{x}) &= (F_1(\mathbf{x}),\ \dots\ , F_m(\mathbf{x})) \\ &= (F_1(x_1,\dots,x_n),\dots,F_m(x_1,\dots,x_n)). \end{aligned}$$

We want to say what it means for F to be continuous. The definition is the same as that for an ordinary function, and is best explained using limits:

Let $\mathbf{a} \in \mathbb{R}^n$. We say that $\lim\limits_{\mathbf{x}\to\mathbf{a}} F(\mathbf{x})$ exists if there exists $\underline{\ell} \in \mathbb{R}^m$ such that "$F(\mathbf{x})$ *approaches* $\underline{\ell}$ *whenever* $\mathbf{x}$ *approaches* $\mathbf{a}$."
This means: given $\frac{1}{10}$, there exists $\delta > 0$ such that

$$0 < \|\mathbf{x}-\mathbf{a}\| < \delta \quad \Rightarrow \quad \|F(\mathbf{x}) - \underline{\ell}\| < \tfrac{1}{10}.$$

(It is important to note that we are not allowed to test $\mathbf{x} = \mathbf{a}$).
Now replace $\frac{1}{10}$ by an arbitrary $\varepsilon > 0$!

It may be that $\mathbf{a} \notin D$ so that $F(\mathbf{a})$ is not defined, but if it is:

Definition. $F\colon \mathbb{R}^n \supseteq D \to \mathbb{R}^m$ is ***continuous*** at $\mathbf{a} \in D$ if

$$\lim_{\mathbf{x}\to\mathbf{a}} F(\mathbf{x}) \text{ exists and equals } F(\mathbf{a}).$$

Equivalently, given $\varepsilon > 0$, $\exists \delta > 0$ such that

$$\|\mathbf{x} - \mathbf{a}\| < \delta \quad \Rightarrow \quad \|F(\mathbf{x}) - F(\mathbf{a})\| < \varepsilon.$$

Since $\|F(\mathbf{x})\|^2 = \sum |F_k(\mathbf{x})|^2$, F will be continuous if and only if the components $F_1, \ldots, F_m$ are all continuous. Thus, we only considered examples with $m = 1$ and $n = 2$:

Let $D = \mathbb{R}^2 \setminus \{(0,0)\}$. Then we showed that

$f(x,y) = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta$ is not continuous at $\mathbf{a} = (0,0)$.

$G(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{xy}{\sqrt{x^2+y^2}} & \text{otherwise} \end{cases}$ is continous at $(0,0)$.

One can appreciate the definition of continuity with the

Proposition. Any linear mapping $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous at every point of its domain $D = \mathbb{R}^n$!

Proof. Let $A \in \mathbb{R}^{m,n}$ be the matrix associated to f (relative to the canonical bases) and let $\mathbf{x}, \mathbf{a} \in \mathbb{R}^{n,1}$. Then

$$\|f(\mathbf{x}) - f(\mathbf{a})\| = \|A\mathbf{x} - A\mathbf{a}\| = \|A(\mathbf{x} - \mathbf{a})\|.$$

It can be shown (not easily even if $m = n = 2$) that the last term is less than or equal to $\|A\|\|\mathbf{x} - \mathbf{a}\|$ where $\|A\| = \sqrt{\sum a_{ij}{}^2}$. Anyway,

$$\|\mathbf{x} - \mathbf{a}\| < \frac{1}{10\|A\|} \quad \Rightarrow \quad \|f(\mathbf{x}) - f(\mathbf{a})\| < \frac{1}{10}$$

(we can assume that A is not the zero matrix so $\|A\| \neq 0$), and continuity follows replacing $\frac{1}{10}$ by an arbitrary $\varepsilon > 0$.