Line integrals along curves 23/05

Let $\gamma(t) = (x(t), y(t), z(t))$ be a space curve.

Its *speed*

$$\frac{ds}{dt} = \|\gamma'(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

represents the rate of change of *arc length* s measured along the curve.

We can express s as a function of t by integrating:

$$s(t) = \int \|\gamma'(t)\|\, dt \ + \text{constant}.$$

If we fix the start $\gamma(a)$ and the end $\gamma(b)$ of the curve, the *length* of the curve equals

$$\ell(\gamma) = s(b) - s(a) = \int_a^b \|\gamma'(t)\|\, dt.$$

If $\|\gamma'(t)\| = 1$ then $s = t + c$ and the curve has *unit speed*. In this case (or if $s = -t + c$), $\gamma'(t)$ is a *unit tangent vector*.

Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a continuous function. It defines a function $f \circ \gamma$ on the curve by composition:

$$f(t) = f(\gamma(t)) = f(x(t), y(t), z(t)).$$

Definition. The *line integral* of f along the curve is

$$\int_\gamma f = \int_a^b f(t)\, \|\gamma'(t)\|\, dt.$$

Up to sign, this is independent of the way in which the curve is parametrized, and in theory we can always use arc length:

$$\int_\gamma f = \int_{s(a)}^{s(b)} f(s)\, ds.$$

The length $\ell(\gamma)$ is the integral of the constant function 1.