Orthogonal matrices and quadratic forms 13/05

SUMMARY

A matrix $P \in \mathbb{R}^{n,n}$ is orthogonal if and only if its columns (or, equivalently, its rows) form an orthonormal basis of $\mathbb{R}^n$. In this case,

$$\det P = \pm 1,$$

and if $n = 2$ or 3 and $\det P = 1$, then P represents a *rotation*.

If $S \in \mathbb{R}^{n,n}$ is symmetric, the function $f\colon \mathbb{R}^n \to \mathbb{R}$ defined by $f(\mathbf{v}) = \mathbf{v} \cdot (S\mathbf{v})$ is called a *quadratic form*. For example

$$f(x, y) = (x\ y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2.$$

Diagonalizing S by a rotation matrix P, we obtain

$$f(x, y) = \lambda_1 X^2 + \lambda_2 Y^2$$

where λ_1, λ_2 are the eigenvalues of S and

$$\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X\cos\theta - Y\sin\theta \\ X\sin\theta + Y\cos\theta \end{pmatrix}$$

expresses the old coordinates in terms of the new ones X, Y.