Powers of square matrices 15/03

SUMMARY

A square matrix $A \in \mathbb{R}^{n,n}$ is ***diagonal*** if $i \neq j \ \Rightarrow \ a_{ij} = 0$.
A particular case is the ***identity matrix*** I_n with entries δ_{ij} (meaning 1 if $i = j$, and 0 otherwise).

If $B \in \mathbb{R}^{m,n}$ then $I_m B = B = B I_n$.

If $A, B, C \in \mathbb{R}^{n,n}$ then both AB and BA are defined in $\mathbb{R}^{n,n}$, but they are not in general equal. Matrix addition and multiplication are both ***associative***:

$$(A + B) + C = A + (B + C), \qquad (AB)C = A(BC).$$

The second formula is not straightforward.

If m, n are positive integers then A^m and A^n are defined in the obvious way ($A^1 = A,\ A^2 = AA, \dots$ etc), and

$$A^{m+n} = A^m A^n = A^n A^m.$$

This remains true if one or both of m, n is 0 once we define A^0 to be the identity matrix. We can extend it further by the

Definition. Given $A \in \mathbb{R}^{n,n}$, its ***inverse*** is a matrix denoted A^{-1} (if it exists) such that $A A^{-1} = I = A^{-1} A$.

A 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $ad - bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The condition $ad - bc$ is equivalent to asserting that there is a ***linear relation***

$$a \begin{pmatrix} b \\ d \end{pmatrix} - b \begin{pmatrix} a \\ c \end{pmatrix} = 0$$

between the columns of A (and a similar one between rows).