Eigenvectors and eigenvalues 03/05

Summary

Given an endomorphism $f\colon V \to V$, we seek elements **v** of the vector space V such that

$$f(\mathbf{v}) = \lambda\mathbf{v}, \qquad \lambda \in F.$$

Such a **v** called an ***eigenvector*** of f, and the scalar λ the associated eigenvalue. (Note that $\mathbf{v} \neq 0$, but λ could be 0.)

An example of a linear map $\mathbb{R}^3 \to \mathbb{R}^3$ that always has an eigenvector with eigenvalue $\lambda = 1$ is a rotation about the origin.

If $V = \mathbb{R}^{n,1}$ consists of column vectors, and f is represented by a square matrix $A \in \mathbb{R}^{n,n}$, the equation becomes

$$A\mathbf{v} = \lambda\mathbf{v} \qquad \text{or} \qquad (A - \lambda I)\mathbf{v} = \mathbf{0}.$$

So **v** is an eigenvector of A if and only if $\mathbf{v} \in \mathrm{Ker}(A - \lambda I)$.

It follows that the possible eigenvalues are the roots of the ***characteristic polynomial***

$$p(x) = \det(A - xI) = (-1)^n x^n + \cdots$$

Having found a root λ, the associated eigenvectors are the non-zero solutions $\mathbf{v} = X$ of the homogeneous linear system

$$(A - \lambda I)X = \mathbf{0}.$$