The sum of two subspaces 02/05

SUMMARY

Let W be a vector space. A *subspace* of W is defined as for $\mathbb{R}^n$:

(S1) $\mathbf{u}, \mathbf{v} \in S \;\Rightarrow\; \mathbf{u}+\mathbf{v} \in S$;

(S2) $a \in F,\ \mathbf{v} \in S \;\Rightarrow\; a\mathbf{v} \in S$.

Any subspace must contain the zero vector 0.

Examples of subspaces: Let $f\colon V \to W$ be a linear mapping.

- $\operatorname{Ker} f$ **is a subspace of** V;
- $\operatorname{Im} f$ **is a subspace of** W.

Their dimensions satisfy $\dim V = \dim(\operatorname{Ker} f) + \dim(\operatorname{Im} f)$.

Now let U, V be subspaces of a vector space W.

- **Their intersection $U \cap V$ is always subspace, possibly $\{\mathbf{0}\}$.**
- **The union $U \cup V$ is only a subspace if it equals U or V.**

The smallest subspace of W containing $U \cup V$ equals

$$\{\mathbf{u}+\mathbf{v} : \ \mathbf{u} \in U \text{ and } \mathbf{v} \in V\},$$

and is denoted $U + V$. It is called the *sum* of U and V, and is the subspace 'generated' by U and V.

If $U = \mathscr{L}\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ and $V = \mathscr{L}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ then

$$U + V = \mathscr{L}\{\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{v}_1, \dots, \mathbf{v}_n\},$$

but these $m+n$ elements may not be LI, even if separately they are. In fact, $\dim(U+V) = \dim U + \dim V - \dim(U \cap V)$.