Linear maps, matrices, systems 19/04

SUMMARY

Let V be a vector space with a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Any linear mapping $f: V \to W$ is determined by the n vectors

$$f(\mathbf{v}_1),\ \ldots\ , f(\mathbf{v}_n)\ \in\ W.$$

If W has a basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ we can write

$$f(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + \cdots + a_{mj}\mathbf{w}_m = \sum_{i=1}^{m} a_{ij}\mathbf{w}_i.$$

The coefficients determine the jth column of the matrix A ***associated*** to the linear mapping f.

Let $f: V \to W$ be linear mapping between two vector spaces. Then $\mathrm{Ker}\, f$ is a subspace of V and $\mathrm{Im}\, f$ is a subspace of W.

Rank-nullity theorem: $\dim V = \dim(\mathrm{Ker}\, f) + \dim(\mathrm{Im}\, f)$.

To prove this, we choose bases and convert f into a matrix $A \in \mathbb{R}^{m,n}$. We may then pretend that $f: \mathbb{R}^n \to \mathbb{R}^m$ is given by $f(X) = AX$.

We know that $\mathrm{Ker}\, f = \mathrm{Ker}\, A$ has dimension $n - r$ where $r = \mathrm{rank}\, A$. But $\mathrm{Im}\, f = \mathrm{Col}\, A$ is the subspace of $\mathbb{R}^m$ spanned by the ***columns*** of A, and has dimension r.