Linear mappings 14/04

SUMMARY

Any matrix $A \in \mathbb{R}^{m,n}$ **defines a linear mapping**

$$f\colon\ \mathbb{R}^{n,1} \longrightarrow \mathbb{R}^{m,1}$$

between spaces of column vectors by setting $f(X) = AX$**. We can regard** f **as a 'function' from** $\mathbb{R}^n$ **to** $\mathbb{R}^m$**. For example, when** $m = n = 2$**, we obtain**

$$f(x, y) = (ax + by,\ cx + dy).$$

Any linear mapping $f\colon V \to W$ **between vector spaces has the property** $f(\mathbf{0}) = \mathbf{0}$**. Its *kernel***

$$\ker f = \{\mathbf{v} \in V : f(\mathbf{v}) = \mathbf{0}\} = f^{-1}(\mathbf{0})$$

is actually a subspace of V **and equals** $\{\mathbf{0}\}$ **iff** f **is injective. When** f **is defined by a matrix** A **as above,** $\ker f = \ker A$ **is the space of solutions of the associated homogeneous linear system** $AX = \mathbf{0}$**.**

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ **is a basis of** V **then the *linear* mapping**

$$\begin{array}{rccc} f\colon & \mathbb{R}^n & \longrightarrow & V \\ & (a_1, \ldots, a_n) & \mapsto & a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n \end{array}$$

is both *sur*jective (B1) and *in*jective (B2). It is an *isomorphism* that can be used to treat V **as the *same* vector space as** $\mathbb{R}^n$ **with respect to the chosen basis.**