More bases and subspaces 12/04

SUMMARY

A vector space V is ***finite-dimensional*** if there exist $\mathbf{u}_1, \ldots, \mathbf{u}_k$ such that $V = \mathscr{L}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$. If this is true we can extract a (possibly smaller) set

$$\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq \mathscr{L}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$$

whose elements are LI and therefore form a ***basis*** of V. Any other basis will have n elements, and $\dim V = n$.

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of V then every element of V can be written ***uniquely*** as a LC

$$\mathbf{v} = a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n, \qquad a_i \in F,$$

and the scalars $a_1, \ldots, a_n$ are called the ***components*** of $\mathbf{v}$ with respect to the basis.

Any ***subspace*** of V (a subset closed under addition and scalar multiplication) is a vector space in its own right. For example $V = \mathbb{R}^{n,n}$ is a vector space of dimension n^2 whilst the set S of symmetric $n \times n$ matrices is a subspace of dimension $\frac{1}{2}n(n+1)$.

Definition. Given two vector spaces V, W with the same field F, a mapping $f: V \to W$ is called ***linear*** if

(LM1) $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$

(LM2) $f(a\mathbf{v}) = a\,f(\mathbf{v})$

for all $\mathbf{u}, \mathbf{v} \in V$ and $a \in F$.

We shall first use such mappings to identify different vector spaces of the same dimension.