A VECTOR SPACE consists of

- a set or *space* V of 'vectors' (arrows, forces, columns, rows, matrices, polynomials, functions...) including $\mathbf{0}$
- a set or *field* F of scalars (usually $F = \mathbb{R}$ in this course) including 0 and 1

such that one can

- add two vectors: $\mathbf{u}, \mathbf{v} \in V \Rightarrow \mathbf{u} + \mathbf{v} \in V$
- multiply a vector by a scalar: $a \in F,\ \mathbf{v} \in V \Rightarrow a\mathbf{v} \in V$

with the following rules:

V
$$\boxed{\begin{aligned} (\mathbf{u}+\mathbf{v})+\mathbf{w} &= \mathbf{u}+(\mathbf{v}+\mathbf{w}) \\ \mathbf{u}+\mathbf{v} &= \mathbf{v}+\mathbf{u} \\ \mathbf{0}+\mathbf{v} &= \mathbf{v} \end{aligned}}$$
for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$

FV
$$\boxed{\begin{aligned} (a+b)\mathbf{v} &= a\mathbf{v}+b\mathbf{v} \\ (ab)\mathbf{v} &= a(b\mathbf{v}) \\ a(\mathbf{u}+\mathbf{v}) &= a\mathbf{u}+a\mathbf{v} \\ 1\mathbf{v} &= \mathbf{v} \\ 0\mathbf{v} &= \mathbf{0} \end{aligned}}$$
for all $a, b \in F,\ \ \mathbf{u}, \mathbf{v} \in V$

Elements of F (if not $\mathbb{R}$) must obey similar rules:

F
$$\boxed{\begin{array}{ccc} (a+b)+c = a+(b+c) & & (ab)c = a(bc) \\ a+b = b+a & & ab = ba \\ 0a = 0 & & 1a = a \\ & ab^{-1} = a/b \quad \text{exists} \quad \text{if } b \neq 0 & \end{array}}$$

Alternative fields include $\mathbb{C}$, $\mathbb{Q}$, $B = \mathbb{F}_2$, $\mathbb{F}_{p^k}$

EXAMPLES of real vector spaces

$F = \mathbb{R}$ and V is one of:

- the set $\mathbb{R}^n$ of vectors with n components
- more precisely, $\mathbb{R}^{1,n}$ or $\mathbb{R}^{n,1}$
- any *subspace* of these spaces, as previously defined
- the set $\mathbb{R}^{m,n}$ of $m \times n$ matrices
- the set $\mathbb{R}[x]$ of polnomials with real coefficients
- the set $\mathbb{R}_d[x]$ of such polynomials with degree $\leqslant d$.

The spaces $\mathbb{R}^{m,n}$ and $\mathbb{R}^{1,mn}$ share the same operations of addition and scalar multiplication once their elements are identified (but there are many valid choices).

We can try to define a finite *basis* of a vector space using (B1),(B2) as before. The *dimension* of a vector space is the number of elements in any such basis, and can be shown not to depend on the choice of basis.

The space $\mathbb{R}^{m,n}$ has dimension mn.

We can define a *subspace* of any vector space V using (S1),(S2) as before.

$V = \mathbb{R}[x]$ does not have finite dimension, but $\mathbb{R}_d[x]$ is a subspace of V of dimension $d + 1$.