Dimension and rank 07/04

SUMMARY (IN REVERSE ORDER TO THE LECTURE!)

Given a matrix $A \in \mathbb{R}^{m,n}$, define its *kernel* or *null space*

$$\operatorname{Ker} A = \{X \in \mathbb{R}^{n,1} : AX = \mathbf{0}\},$$

consisting of the solutions to the homogeneous system. The Gauss-Jordan method will give us one basis element for each free parameter. The dimension of a subspace is the number of elements in a basis, so $\dim(\operatorname{Ker} A) = n - r$ where r is the rank of A.

If $R_1, \ldots, R_m$ are the rows of A, then its *row space*

$$\operatorname{Row} A = \mathscr{L}\{R_1, \ldots, R_m\} \subseteq \mathbb{R}^{1,n}$$

is the subspace generated by the rows. A basis consists of the non-zero rows in any step-reduced version of A, so $\dim(\operatorname{Row} A) = r$. This gives us a way of finding a basis of any subspace generated by a set of vectors, better than 'trashing dependents'.

Elements of $\operatorname{Ker} A$ are columns, and those of $\operatorname{Row} A$ rows, but both can be regarded as subspaces of $\mathbb{R}^n$. The methods we have developed can be used to prove the

Theorem. Given two matrices A, A' of the same size,

$$\begin{aligned} A \sim A' \quad &\Leftrightarrow \quad \operatorname{Ker} A = \operatorname{Ker} A' \\ &\Leftrightarrow \quad \operatorname{Row} A = \operatorname{Row} A'. \end{aligned}$$

Any subspace can be defined as $\operatorname{Row} A$ for some A, and one can use this to prove that the dimension of a subspace of $\mathbb{R}^n$ is at most n and independent of the choice of basis.