1. The matrix $\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & -1\\ 1 & -1\end{pmatrix}$ represents a rotation of the xy axes by an angle of
   (a) $\pi/4$, (b) $\pi/2$, (c) $3\pi/4$, (d) π, (e) $5\pi/4$.

2. The inverse of the matrix $\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$ is
   (a) itself, (b) $\begin{pmatrix}1 & -1 & -1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}$, (c) $\begin{pmatrix}1 & -1 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, (d) $\begin{pmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}$,
   (e) none of the above.

3. Consider, in $\mathbb{R}^4$, the subspaces

$$V = \mathscr{L}\left\{(1,2,1,0),\ (2,1,0,1)\right\}, \qquad W = \mathscr{L}\left\{(0,3,2,-5),\ (1,-1,-1,1)\right\}.$$

   Their sum $V + W$ is a subspace of dimension
   (a) 4, (b) 3, (c) 2, (d) 1, (e) 0.

4. A linear mapping $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $f(x,y,z) = (x+z,\ 2y,\ x+y+z)$. Then
   (a) there exists a non-zero vector $\mathbf{v} = (x,y,z)$ such that $f(\mathbf{v}) = \mathbf{v}$,
   (b) f is bijective, (c) $f^{-1}(\mathbf{0}) = \{\mathbf{0}\}$, (d) $f^{-1}(\mathbf{0})$ is infinite.
   (e) none of the above.

5. The matrix $\begin{pmatrix}0 & a & c\\ 0 & 0 & b\\ 0 & 0 & 1\end{pmatrix}$ is diagonalizable if and only if
   (a) $a = 0$, (b) $b = 0$, (c) $c = 0$, (d) $a = b = c = 0$,
   (e) none of the above.

6. The two lines

$$r_1 : \begin{cases} x + y + z = 1 \\ x + 2y + 3z = 1, \end{cases} \qquad r_2 :\ (x,y,z) = (t+1,\ 1-2t,\ t), \quad t \in \mathbb{R},$$

   (a) meet in one point, (b) are parallel, (c) are orthogonal, (d) are skew,
   (e) none of the above.

A Let $A = \begin{pmatrix} -2 & 2 & 3 \\ 2 & -5 & 2 \\ 4 & 2 & -1 \end{pmatrix}$.

(a) Show that

$$\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 7 \\ 4 \\ 9 \end{pmatrix}$$

are both eigenvectors of A. [Multiply each in turn by A.] What are their associated eigenvalues?

(b) What does your answer to part (a) tell you about the characteristic polynomial $p(x)$ of A? Determine $p(x)$ completely, or say what its roots are. [Recall that the sum of the roots is the *trace* of A.]

(c) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Write down the determinant of D and of A.

Solution:

B Consider the matrix and column vectors

$$A = \begin{pmatrix} 1 & -1 & 0 & -2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 4 & 0 \\ 0 & 2 & 5 & 0 \end{pmatrix}, \qquad X = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}, \qquad B = \begin{pmatrix} 3 \\ 2 \\ 1 \\ -2 \end{pmatrix}.$$

(a) Row reduce the augmented matrix $(A\,|\,B)$, and hence find all solutions of the linear system $AX = B$.

(b) Using your answer to part (a), write down the dimensions of the kernel and image of the linear mapping $\mathbb{R}^4 \to \mathbb{R}^4$ associated to A.

(c) Using your answer to (a), find a non-zero column vector X such that $AX = \mathbf{0}$, and describe the kernel of A.

(d) Using your answer to (c), find all solutions X to the equation $AX = AB$.

Solution:

C Let $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, $\mathbf{k} = (0,0,1)$. Attempt the following short questions:

(a) Describe the set of points (x, y, z) is space for which the corresponding vector $\mathbf{v} = (x, y, z)$ satisfies $\mathbf{v} \cdot \mathbf{i} = 1$.

(b) Describe the set of points (x, y, z) in space for which $\mathbf{v} = (x, y, z)$ satisfies the equation $\mathbf{v} \times \mathbf{j} = \mathbf{k}$.

(c) Decide whether the three vectors $\mathbf{j} + \mathbf{k}$, $2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, $8\mathbf{i} + 13\mathbf{j} + 21\mathbf{k}$ are linearly independent.

(d) Show that the set of points (x, y) in the plane satisfying the equation

$$\left(x\mathbf{i} - 2y\mathbf{j}\right) \times \left((2x + y)\mathbf{i} + 3x\mathbf{j}\right) = \mathbf{k}$$

is a conic, and say whether it is an ellipse, hyperbola or something else.

[Recall that:

the scalar product of two vectors $(a, b, c) \cdot (x, y, z)$ of two vectors equals $ax + by + cz$;

the cross product $(a, b, c) \times (x, y, z)$ of two vectors equals $(bz - cy,\ cx - az,\ ay - bx)$.]

Solution: