1. Consider the two lines defined parametrically by r_1: $(x, y, z) = (s+1,\, s,\, -s)$ and r_2: $(x, y, z) = (t, t, t)$. Then
   (a) the directions of r_1 and r_2 are orthogonal,
   (b) r_1 and r_2 are parallel,
   (c) r_1 and r_2 are skew,
   (d) r_1 and r_2 meet in one point.

2. Consider the function $f(x, y) = e^{x^2+y^2-1}$. Then
   (a) f has no critical points,
   (b) the tangent plane to the graph of f at $(x, y) = (0, 0)$ is $z = e^{-1} + x + y$,
   (c) f has a saddle point at $(x, y) = (0, 0)$,
   (d) f has a minimum at $(x, y) = (0, 0)$.

3. Consider the sphere $\mathscr{S}$ with equation $x^2 + y^2 + z^2 + 2x + 2y + 2z = 0$. Let r be its radius and C its centre. Then
   (a) $r = 1$,
   (b) r equals the distance of C from the plane $x + y + x = 0$,
   (c) $C = (1, 1, 1)$,
   (d) $C = (0, 0, 0)$.

4. Consider the linear mapping $f\colon \mathbb{R}^4 \to \mathbb{R}^2$ with $f(x, y, z, t) = (x - y + t,\ x + t)$. Then
   (a) the matrix associated to f has 4 rows and 2 columns,
   (b) f is surjective,
   (c) the image of f has dimension 1,
   (d) the kernel of f has dimension 1.

5. Let π be the plane that passes through $(1, 1, 1)$ perpendicular to $\mathbf{i} + \mathbf{j} + \mathbf{k}$. Then
   (a) the line $(x, y, z) = (t, 0, -t)$ is parallel to π,
   (b) the line $(x, y, z) = (t, 0, -t)$ intersects π,
   (c) the line $(x, y, z) = (t, t, t)$ is parallel to π,
   (d) $(1, 0, -1)$ lies on π.

6. Consider the curve $\gamma(t) = (\cos t,\, \sin t,\, t)$. The arc length from $t = 0$ to $t = 2\pi$ (one twist of the helix) equals
   (a) 2π,
   (b) 3π,
   (c) $2\pi\sqrt{2}$,
   (d) $2\pi + 3$.

7. Let $f(x,y,z) = x^2 + 2y + z$. Let $\widehat{\nabla F} = (\nabla F)(1,1,1)$ and let $\mathbf{v}$ be a unit vector (so $\|\mathbf{v}\| = 1$). The directional derivative $\mathbf{v} \cdot \widehat{\nabla F}$ has its greatest value when
   (a) $\mathbf{v} = \frac{1}{3}(2,2,1)$,
   (b) $\mathbf{v} = (1,0,0)$,
   (c) $\mathbf{v} = \frac{1}{\sqrt{3}}(1,1,1)$,
   (d) $\mathbf{v} = (0,0,1)$.

8. Consider the quadric $\mathscr{Q}$ in $\mathbb{R}^3$ with equation $z^2 = xy$. Then
   (a) $\mathscr{Q}$ is a hyperbolic paraboloid,
   (b) $\mathscr{Q}$ is a cone,
   (c) $\mathscr{Q}$ is a hyperboloid of one sheet,
   (d) $\mathscr{Q}$ is a hyperboloid of two sheets.

9. Consider the matrices $A = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$, $B = (1 \quad 1 \quad 1)$ and $C = AB \in \mathbb{R}^{3,3}$. Then
   (a) C is symmetric,
   (b) the trace of C is positive,
   (c) C^2 is the zero matrix,
   (d) $\det C$ is negative.

10. Let $A \in \mathbb{R}^{3,4}$ be a matrix (with 3 rows and 4 columns) of rank 3, and let $X \in \mathbb{R}^{4,1}$ be a column vector. Then
   (a) the linear system $AX = B$ always has a unique solution,
   (b) there exists B such that $AX = B$ has no solutions,
   (c) solutions of the homogeneous system $AX = \mathbf{0}$ have 2 free parameters,
   (d) the linear system $AX = B$ always has infinitely many solutions.

11. Consider the column vectors $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 9 \end{pmatrix}$. Then
   (a) $\mathbf{v}_1 \times \mathbf{v}_2$ and $\mathbf{v}_3$ are parallel,
   (b) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbb{R}^3$,
   (c) the triple product $(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3$ is zero,
   (d) $\mathbf{v}_1 + \mathbf{v}_2$ is orthogonal to $\mathbf{v}_3$.

12. Consider the conic $\mathscr{C} = \{(x,y) \in \mathbb{R}^2 : x^2 - 3xy + 8y^2 = 4\}$ in the plane. Then
   (a) $\mathscr{C}$ is an ellipse,
   (b) $\mathscr{C}$ is a hyperbola,
   (c) $\mathscr{C}$ consists of two intersecting lines,
   (d) $\mathscr{C}$ is the union of two parallel lines.

A Consider the following two subspaces of $\mathbb{R}^3$:

$$U = \mathscr{L}\{(1,0,2,3),\ (1,-6,6,5),\ (2,3,2,5)\}$$
$$V = \mathscr{L}\{(1,-2,2,3),\ (1,-3,4,4),\ (2,-1,-2,3)\}.$$

(i) Find the dimension of U.

(ii) Determine the dimension of V, and write down a basis of V.

Now consider the subspace $U+V$ generated by all six vectors.

(iii) Determine the dimension of $U+V$.

(iv) Deduce (with a brief explanation) that $\dim(U\cap V)=1$.

B Consider the matrix

$$A = \begin{pmatrix} 3 & 4 \\ 4 & 18 \end{pmatrix}.$$

(i) Find the eigenvalues of A.

(ii) Find two linearly independent eigenvectors of A.

(iii) Find an *orthogonal* matrix P and a diagonal matrix D such that $P^{-1}AP = D$. (There is no need to verify this equation.)

(iv) Let P^T denote the transpose of P. Explain why $A = PDP^T$ and also $A^5 = PD^5P^T$. (It is not necessary to multiply the matrices numerically!)

C Consider the functions

$$f(x,y) = x^3 + y^3 - xy, \qquad F(x,y,z) = z - f(x,y),$$

so that the graph of f is the surface $F=0$.

(i) Compute the gradient of f, and also the gradient of F.

(ii) Verify that $F(1,1,1)=0$ and write down the equation of the tangent plane to the graph of f at $(1,1,1)$.

(iii) Find the critical points of f (showing your working). Classify the type of each critical point (minimum/maximum/saddle).

Now let $\gamma\colon [0,1] \to \mathbb{R}^3$ be the line segment with $\gamma(t) = (0,1,2t)$.

(iv) Find $\|\gamma'(t)\|$ and compute the line integral $\displaystyle\int_\gamma F$.