1. Consider the subset $S = \{(x, y, z) \in \mathbb{R}^3 : x + y - 1 = 0 \text{ and } z = 0\}$ of $\mathbb{R}^3$. Then
   (a) S is a subspace of dimension 1,
   (b) S is a subspace of dimension 2,
   (c) S does not contain more than two linearly independent vectors,
   (d) S contains a basis of $\mathbb{R}^3$.

2. Consider the vectors $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{w} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$ in $\mathbb{R}^3$. Then
   (a) $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis of $\mathbb{R}^3$,
   (b) $\mathbf{w}$ is parallel to the cross product $\mathbf{u} \times \mathbf{v}$,
   (c) $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are coplanar (that is, not linearly independent),
   (d) $\mathbf{v}$ is parallel to the cross product $\mathbf{u} \times \mathbf{w}$.

3. Consider the function of two variables $f(x, y) = 1 + x + y$. Then
   (a) the point $(1, 1, 1)$ belongs to the graph of f,
   (b) $(\nabla f)(0, 0) = (\nabla f)(1, 1)$,
   (c) f is not differentiable at the origin,
   (d) $f(x, y) \geqslant 1$ for all $(x, y) \in \mathbb{R}^2$.

4. Consider the matrices $A = (2 \quad -2 \quad 1)$ and $B = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. Then
   (a) the rank of BA equals 2,
   (b) 0 is an eigenvalue of BA,
   (c) BA is symmetric,
   (d) BA is invertibile.

5. Let $f(x, y) = e^{x+y^2}$. Then
   (a) the Taylor expansion to first order of f at $(0, 0)$ is x,
   (b) the Taylor expansion to first order of f at $(0, 0)$ is $1 + x + y^2$,
   (c) the Taylor expansion to first order of f at $(0, 0)$ is $x + y^2$,
   (d) the Taylor expansion to first order of f at $(0, 0)$ is $1 + x$.

6. Let $\mathscr{C}$ be the conic with equation $-x^2 + y^2 + 2x + 3 = 0$ in the plane $\mathbb{R}^2$. Then
   (a) $\mathscr{C}$ consists of two intersecting lines,
   (b) $\mathscr{C}$ is a hyperbola,
   (c) $\mathscr{C} = \varnothing$ (that is, $\mathscr{C}$ contains no real points),
   (d) $\mathscr{C}$ is a parabola.

7. Let $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then
   (a) N is diagonalizable,
   (b) the characteristic polynomial $p(x)$ of N equals $-x^3$,
   (c) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of N,
   (d) the characteristic polynomial $p(x)$ of N equals $x^2 + 1$.

8. Let $\mathscr{S}$ be the surface parametrized by $\sigma(s,t) = (s,\ t,\ st)$. Then
   (a) a normal vector to $\mathscr{S}$ at $(1,1,1)$ is $(1,1,1)$,
   (b) a normal vector to $\mathscr{S}$ at $(1,1,1)$ is $(-1,-1,1)$,
   (c) a normal vector to $\mathscr{S}$ at $(1,1,1)$ is $(1,-1,1)$,
   (d) a normal vector to $\mathscr{S}$ at $(1,1,1)$ is $(-1,1,1)$.

9. Consider the vectors $\mathbf{v}_1 = (0,1,0,0)$, $\mathbf{v}_2 = (0,1,0,1)$, $\mathbf{v}_3 = (1,-2,0,2)$ in $\mathbb{R}^4$. Then
   (a) $\mathscr{L}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ has dimension 4,
   (b) $\mathscr{L}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ has dimension 2,
   (c) $\mathbf{v}_1 - \mathbf{v}_2 + 5\mathbf{v}_3 \in \mathscr{L}\{\mathbf{v}_1, \mathbf{v}_2\}$,
   (d) there exists $\mathbf{v}_4$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis of $\mathbb{R}^4$.

10. Consider the quadric $\mathscr{Q}$ with equation $x^2 + y^2 = 1$ in space $\mathbb{R}^3$. Then
    (a) $\mathscr{Q}$ is a paraboloid,
    (b) $\mathscr{Q}$ has points that lie on the plane $z = 1$,
    (c) $\mathscr{Q}$ is a circle of radius 1,
    (d) $\mathscr{Q}$ is a cone.

11. Consider the linear mapping $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by
    $$f(x,y,z) = (y + z,\ 2y + 2z,\ 3y + 3z).$$
    Then
    (a) the kernel $\operatorname{Ker}(f)$ has dimension 2,
    (b) f is injective,
    (c) the image $\operatorname{Im}(f)$ has dimension 2,
    (d) $(1,1,1)$ belongs to $\operatorname{Im}(f)$.

12. Consider the curve $\mathscr{C}$ in $\mathbb{R}^3$ with parametric equation $\gamma(t) = (\sin t,\ 2\cos t,\ t^2)$. Then
    (a) $\mathscr{C}$ lies in a plane,
    (b) there exists $t \in \mathbb{R}$ such that $\gamma'(t) = \mathbf{0}$,
    (c) $\mathscr{C}$ lies on a sphere,
    (d) $(0,2,0)$ belongs to $\mathscr{C}$.

A Consider the following two lines in space defined parametrically:

$$r_1\colon\ (x,y,z) = (2t,\ 3t,\ t)$$
$$r_2\colon\ (x,y,z) = (\ 3s+5,\ 3s,\ -s).$$

(i) Show that r_1 and r_2 are skew lines, and that r_1 contains $(0,0,0)$.
(ii) Find the equation of the plane containing r_1 and passing through $(-1,0,1)$.
(iii) Write down the equations of any two planes whose intersection is r_2.

B Consider the matrix

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix},$$

and the linear mapping $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(X) = AX$.
(i) Find a basis of the subspace $\operatorname{Ker} f$.
(ii) Decide whether the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ belongs to $\operatorname{Im}(f)$, justifying your answer.
(iii) Say whether A is invertible, justifying your answer.
(iv) Determine all the eigenvalues of A. Say whether A is diagonalizable, justifying your answer.

C Consider the function of two variables

$$f(x,y) = 2y^3 + (y-x)^2 - 6x.$$

(i) Compute the gradient of f and show that f has two critical points.
(ii) Find the type (for example, local maximum, local minimum, saddle point) of each critical point.

Consider the graph of f, the surface $\mathscr{S}$ with equation $z = 2y^3 + (y-x)^2 - 6x$.
(iii) Show that $P = (0,1,3)$ belongs to $\mathscr{S}$, and find the equation of the tangent plane to $\mathscr{S}$ at P.
(iv) Write down the parametric equation of the line passing through P in the direction of the normal vector to $\mathscr{S}$.