Notes 9 – Vectors in space

We shall represent points and displacements by vectors and study the scalar product.

L9.1 Cartesian coordinates. A choice of these allows us to label a point P in space by a triple (x, y, z). It is convenient to think of z as representing 'height', and (x, y) the point of the 'horizontal' plane that P projects to. We shall regard the corresponding column vector

$$\mathbf{v} = (x, y, z)^\top = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{OP}$$

as the *position vector* of P relative to the origin $O = (0, 0, 0)$.

A column vector should be thought of as a *displacement* in space represented by direction and length, but free to be applied at any point, not necessarily O. This allows us to visualize the *sum* of two column vectors $\mathbf{v} + \mathbf{w}$ as the displacement determined by the diagonal of a paralellogram with sides $\mathbf{v}$ and $\mathbf{w}$.

Moreover, if $\mathbf{w} = \vec{OQ}$ then

$$\vec{OP} + \vec{PQ} = \vec{OQ} \quad \Rightarrow \quad \vec{PQ} = \mathbf{w} - \mathbf{v},$$

and the displacement from P to Q is represented by the *difference* of vectors, or subtraction.

In coordinates,

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix},$$

and two applications of Pythagoras's theorem tell us that its length is

$$\sqrt{(\sqrt{x^2 + y^2})^2 + z^2} = \sqrt{x^2 + y^2 + z^2}.$$

This quantity is called the *norm* or *magnitude* of the vector $\mathbf{v}$, and denoted $\|\mathbf{v}\|$ or $|\mathbf{v}|$. It is the distance from O to P. Similarly, the distance from P to Q is

$$|PQ| = |\mathbf{v} - \mathbf{w}|.$$

A vector of norm 1 is said to be a *unit* vector.

L9.2 Dot or scalar product. We already defined this in order to multiply matrices. If

$$\mathbf{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

then

$$\mathbf{u} \cdot \mathbf{v} = ax + by + cz = \mathbf{u}^\top \mathbf{v}$$

(ignoring the parentheses that are implicit in the last expression). We see immediately that

$$\mathbf{v} \cdot \mathbf{v} = x^2 + y^2 + z^2 = |\mathbf{v}|^2 \geqslant 0,$$

so the norm squared of $\mathbf{v}$ is the scalar product of $\mathbf{v}$ with itself.

The following properties are easy to verify:

(i) $\mathbf{v}\cdot\mathbf{w} = \mathbf{w}\cdot\mathbf{v}$,

(ii) $(a\mathbf{u}+b\mathbf{v})\cdot\mathbf{w} = a\mathbf{u}\cdot\mathbf{w} + b\mathbf{v}\cdot\mathbf{w}$. Thus,

$$\begin{aligned} |\mathbf{v}-\mathbf{w}|^2 \;=\; (\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) &= \mathbf{v}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{w} - \mathbf{w}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w} \\ &= |\mathbf{v}|^2 - 2\mathbf{v}\cdot\mathbf{w} + |\mathbf{w}|^2. \end{aligned} \tag{1}$$

We record the well-known

Definition. *Let $\mathbf{v},\mathbf{w}$ be nonnull vectors in $\mathbb{R}^3$. If they make an angle of $\pi/2$ in their common plane, we say that they are* perpendicular *or* orthogonal. *If they make an angle of 0 or π they are* parallel.

In the first case, $\mathbf{v}-\mathbf{w}$ is the hypotenuse of a right-angled triangle. and its length squared is $|\mathbf{v}|^2 + |\mathbf{w}|^2$. It follows from (1) that $\mathbf{v}\cdot\mathbf{w} = 0$. Here we have used Pythagoras's theorem, which is generalized by the *Cosine Rule* (*Teorema di Carnot)* for triangles (equally valid in space because all the action takes place in a fixed plane). This states that

$$|PQ|^2 = |OP|^2 + |OQ|^2 - 2|OP||OQ|\cos\theta, \tag{2}$$

where $\theta = \angle POQ$. Comparing (1) and (2) gives the

Lemma. *The scalar product can be expressed by the formula*

$$\mathbf{v}\cdot\mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta,$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

Since $\cos\theta = \cos(-\theta)$, we do not have to worry about the order in which we choose the two vectors to measure the angle; this is consistent with (i) above.

The Lemma confirms that

Corollary. *(i) If $\mathbf{v},\mathbf{w}$ are both nonzero then $\mathbf{v}\cdot\mathbf{w} = 0$ iff $\mathbf{v},\mathbf{w}$ are orthogonal.*
(ii) $|\mathbf{v}\cdot\mathbf{w}| \leqslant |\mathbf{v}||\mathbf{w}|$ with equality iff $\mathbf{v},\mathbf{w}$ are parallel.

L9.3 Orthonormal bases. The canonical base of $\mathbb{R}^{3,1}$ consists of

$$\mathbf{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

These three vectors (or their row counterparts $e_i^\top$) are often denoted $\mathbf{i}$, $\mathbf{j}\,\mathbf{k}$. They are unit vectors, and mutually orthogonal:

$$|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1, \qquad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0.$$

These relations can be neatly expressed by the single formula

$$\mathbf{e}_p\cdot\mathbf{e}_q = \delta_{pq}, \tag{3}$$

where δ_{pq} stands for 1 if $p = q$, and 0 otherwise.

Definition. *A basis* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *of* $\mathbb{R}^3$ *is called* orthonormal *if it satisfies (3) (with* $\mathbf{v}$ *in place of* $\mathbf{e}$*).*

The 'ortho' part means that the vectors are mutually orthogonal, and 'normal' stands for 'normalized', meaning of unit norm.

> **Exercise.** Any set of three distinct vectors that satisfies (3) is necessarily LI, and so automatically a *basis* of $\mathbb{R}^3$.

Given an orthonormal basis, let P denote the matrix whose columns are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Then

$$P^\top P = \begin{pmatrix} \leftarrow \mathbf{v}_1^\top \rightarrow \\ \leftarrow \mathbf{v}_2^\top \rightarrow \\ \leftarrow \mathbf{v}_3^\top \rightarrow \end{pmatrix} \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix} = I_3.$$

Definition. *A square matrix satisfying* $P^\top P = I_n$ *is called* orthogonal.

We shall study such matrices in Part II.

𝔚𝔞𝔯𝔫𝔦𝔫𝔤. The terminology is misleading: the columns of an orthogonal matrix P are not just orthogonal but also orthonormal. It would make more sense to call P an 'orthonormal matrix', but nobody ever does!

L9.4 Components. We have seen that an arbitrary vector $\mathbf{v} \in \mathbb{R}^3$ can be expressed as

$$\mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3.$$

The numbers a_1, a_2, a_3 are called the *coefficients* of $\mathbf{v}$ relative to the canonical basis, and are also given by the formula

$$a_p = \mathbf{v} \cdot \mathbf{e}_p, \qquad p = 1, 2, 3. \tag{4}$$

Exactly the same result holds for any orthonormal basis:

Lemma. *If* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *is an orthonormal basis and* $\mathbf{v}$ *an arbitrary vector then*

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{v}_1)\mathbf{v}_1 + (\mathbf{v}\cdot\mathbf{v}_2)\mathbf{v}_2 + (\mathbf{v}\cdot\mathbf{v}_3)\mathbf{v}_3 = \sum_{p=1}^{3}(\mathbf{v}\cdot\mathbf{v}_p)\mathbf{v}_p$$

Proof. We know that $\mathbf{v}$ can be written uniquely as

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3.$$

Taking the dot product of both sides with (for example) $\mathbf{v}_1$ yields

$$\mathbf{v}\cdot\mathbf{v}_1 = a_1\mathbf{v}_1\cdot\mathbf{v}_1 + a_2\mathbf{v}_2\cdot\mathbf{v}_1 + a_3\mathbf{v}_3\cdot\mathbf{v}_1 = a_1,$$

since $\mathbf{v}_1 \cdot \mathbf{v}_p = \delta_{1p}$. QED

More generally, suppose that $\mathbf{u}$ is a *unit* vector. Then

$$\mathbf{v} = (\mathbf{u}\cdot\mathbf{v})\mathbf{u} + \mathbf{w}, \qquad \mathbf{w} = \mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{u},$$

and (taking its dot product with $\mathbf{u}$) we see that $\mathbf{w}$ is orthogonal to $\mathbf{u}$. Thus, $\mathbf{u}\cdot\mathbf{v}$ is the numerical component of $\mathbf{v}$ in the direction $\mathbf{u}$.

Exercise. Any unit vector $\mathbf{u}\in\mathbb{R}^3$ can be extended to an ON basis $\{\mathbf{u}=\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ of $\mathbb{R}^3$.

L9.5 Further exercises.

1. Let $\mathbf{u} = a\mathbf{i}+2\mathbf{j}+b\mathbf{k}$, $\mathbf{v} = (1-b)\mathbf{i}+b\mathbf{j}+2\mathbf{k}$, $\mathbf{w} = b\mathbf{i}+b\mathbf{j}+2\mathbf{k}$. Find the values of a and b for which the set $\{\mathbf{u}+\mathbf{v},\mathbf{w}\}$ is LI.

2. Let $\mathbf{v} = \mathbf{i}+\sqrt{3}\mathbf{j}+\mathbf{k}$. Is it true that $\mathbf{v}$ mkes an angle of 60^{o} with the x-axis?

3. Let $\mathbf{u}_1 = -\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{u}_2 = \mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{u}_3 = \mathbf{i}+\mathbf{j}-\mathbf{k}$. Which of the followings sets is LI?

$$\{\mathbf{u}_1,\mathbf{u}_2\},\qquad \{\mathbf{u}_1+\mathbf{u}_2,\ \mathbf{u}_1-\mathbf{u}_2\},\qquad \{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\},\qquad \{\mathbf{i},\mathbf{u}_2,\mathbf{u}_3\}.$$

4. Find the angle between the following pairs of vectors

$$\{\mathbf{i},\ \mathbf{i}+\mathbf{j}\},\qquad \{\mathbf{i}+\mathbf{j},\ \mathbf{i}+\mathbf{k}\},\qquad \{\mathbf{i}+\mathbf{j},\ 2\mathbf{i}+\mathbf{j}+\mathbf{k}\}.$$

In each case, find a unit vector perpendicular to the two given vectors.

5. Given $\mathbf{u} = \mathbf{i}+3\mathbf{j}-\mathbf{k}$ and $\mathbf{v} = \mathbf{i}-\mathbf{j}$, decompose $\mathbf{u}$ as the sum of a vector perpendicular to $\mathbf{u}$ and a vector parallel to $\mathbf{v}$.

6. Find all vectors coplanar with $\mathbf{u} = \mathbf{i}-\mathbf{k}$ e $\mathbf{v} = \mathbf{i}+\mathbf{j}$, but orthogonal to $\mathbf{u}+\mathbf{v}$.

7. Compute $|\mathbf{u}+\mathbf{v}|^2$. Deduce that
   (i) $\mathbf{u}$ e $\mathbf{v}$ are orthogonal iff $|\mathbf{u}+\mathbf{v}| = |\mathbf{u}-\mathbf{v}|$,
   (ii) $\mathbf{u}+\mathbf{v}$ e $\mathbf{u}-\mathbf{v}$ are orthogonal iff $|\mathbf{u}| = |\mathbf{v}|$.

8. Find all vectors of norm 5 that are perpendicula to $2\mathbf{i}+\mathbf{j}-3\mathbf{k}$.

9. Consider the vectors

$$\mathbf{v}_1=(2,0,0),\quad \mathbf{v}_2=(1,1,0),\quad \mathbf{v}_3=(3,2,1).$$

Express each in terms of $\mathbf{i},\mathbf{j},\mathbf{k}$ and deduce that $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is a basis of $\mathbb{R}^3$.

10. Given $\mathbf{u} = 5\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i}+\mathbf{j}-2\mathbf{k}$, check that $\mathbf{u} = |\mathbf{v}|\mathbf{i}+\mathbf{v}$. Deduce that $\mathbf{u}$ bisects the angle formed by $\mathbf{v}$ and $\mathbf{i}$.