Notes 8 – Bases and dimension

The concepts of linear combination and linear independence are combined in the definition of the *basis* of a subspace V of $\mathbb{R}^n$. Any two bases have the same number of elements, called the *dimension.* When V is represented as the row space of a matrix A, its dimension equals the rank of A.

L8.1 Redundant elements. Describing a subspace as $\mathscr{L}\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ is all very well, but there are infinitely many ways to choose the representative vectors $\mathbf{u}_1, \cdots, \mathbf{u}_k$. Suppose that $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set, and consider the subspace

$$V = \mathscr{L}\{\mathbf{u},\ 2\mathbf{u},\ \mathbf{u}+\mathbf{v},\ \mathbf{0},\ \mathbf{u}-7\mathbf{v}\}.$$

The right-hand side is a bit ridiculous since most of its elements are redundant. We can make the list of vectors more effective as follows:

Retain the non-zero vector $\mathbf{u}$.
Discard $2\mathbf{u}$ because it is already in $\mathscr{L}\{\mathbf{u}\}$.
Retain $\mathbf{u}+\mathbf{v}$ becuase it is not in $\mathscr{L}\{\mathbf{u}\}$.
Discard $\mathbf{0}$ (since the null vector is already present in $\mathscr{L}\{\mathbf{u}, \mathbf{v}\}$!).
Discard $\mathbf{u}-7\mathbf{v}$ as it too is in $\mathscr{L}\{\mathbf{u}, \mathbf{v}\}$.

We are finished with $V = \mathscr{L}\{\mathbf{u},\ \mathbf{u}+\mathbf{v}\}$, which of course equals $\mathscr{L}\{\mathbf{u}, \mathbf{v}\}$ though it may be that $\mathbf{u}+\mathbf{v}$ is simpler than $\mathbf{v}$ in a numerical example.

Definition. *Let V be a subspace of $\mathbb{R}^n$. A* basis *of V is a linearly independent set $\{\mathbf{v}_1 \cdots, \mathbf{v}_k\}$ such that $V = \mathscr{L}\{\mathbf{v}_1 \cdots, \mathbf{v}_k\}$.*

A basis of V is therefore characterized by two key properties:

(B1) it generates V, meaning that any element in V is a LC of the basis, and
(B2) the basis is LI.

The second condition implies that a basis can contain no more than n elements (for we can represent the basis elements as rows of a matrix with n columns, so at most n rows are LI).

We shall often indicate a particular basis as follows: $\mathscr{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. Some authors require a basis to be an *ordered* set, in which case one could write $\mathscr{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$.

$\mathfrak{Warning}$. Since no LI set can contain $\mathbf{0}$, neither can a basis.

L8.2 Bases of $\mathbb{R}^n$. The definition of basis makes perfect sense if we take the subspace $V = \mathbb{R}^n$ (thought of as either row or column vectors), and there are infintely many bases to choose from. But the most obvious is the one consisting of the rows or columns of the matrix I_n. In particular,

Definition. *The* canonical basis *of $\mathbb{R}^{n,1}$ consists of the columns of I_n, and its individual elements are denoted $\mathbf{e}_1, \ldots, \mathbf{e}_n$.*

Thus,

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \cdot \\ 0 \end{pmatrix}, \quad \ldots$$

It is obvious that any $\mathbf{v} \in \mathbb{R}^{n,1}$ can be written in one and only one way as a LC of this basis:

$$\mathbf{v} = \begin{pmatrix} a_1 \\ \cdot \\ a_n \end{pmatrix} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n.$$

But this property holds for *any* basis of *any* subspace. By property (B1), $\mathbf{v}$ can certainly be written in some way as a LC of the basis, and by (B2),

$$\begin{aligned} \mathbf{v} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n &= b_1\mathbf{e}_1 + \cdots + b_n\mathbf{e}_n \\ \Rightarrow \quad (a_1-b_1)\mathbf{e}_1 + \cdots &+ (a_n-b_n)\mathbf{e}_n = 0 \\ \Rightarrow \quad a_1-b_1 = 0, &\quad \cdots \quad a_n-b_n = 0. \end{aligned}$$

Exercise. Is $\mathscr{B} = \{(2,-1,-1),(1,-2,1),(1,1,-2)\}$ a basis of $\mathbb{R}^{1,3}$?

L8.3 Finding bases by reduction. Bases can in theory be computed by the 'discard/retain' method described above. But often it is more effective to use row reduction. If B is step-reduced, we know that the nonzero rows of B are LI. They certainly generate Row B because the missing rows are null! Thus we have the

Proposition. *The nonzero rows of a step-reduced matrix B form a basis of* Row B.

This gives us a prescription for finding a basis of any susbspace

$$V = \mathscr{L}\{\mathbf{u}_1, \cdots, \mathbf{u}_k\} \subset \mathbb{R}^n.$$

First suppose that the $\mathbf{u}_i$ are row vectors. We can construct a matrix $A \in \mathbb{R}^{k,n}$ by taking these vectors in any order to be the rows of A. Apply ERO's to A to obtain a step-reduced matrix B. Then the nonzero rows of B form a basis of Row B = Row $A = V$.

We can apply an identical procedure if the $\mathbf{u}_i$ are column vectors; we only need to transpose them into rows, and at the end of our labours transpose the nonzero rows of B back into columns. We shall illustrate the latter by constructing a basis of the subspace

$$W = \mathscr{L}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{pmatrix}, \begin{pmatrix} 11 \\ 12 \\ 13 \\ 14 \\ 15 \end{pmatrix}, \begin{pmatrix} 16 \\ 17 \\ 18 \\ 19 \\ 20 \end{pmatrix} \right\} \subset \mathbb{R}^{5,1}.$$

We convert the columns into rows and reduce the 5×4 matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -5 & -10 & -15 & -20 \\ 0 & -10 & -20 & -30 & -40 \\ -15 & -30 & -45 & -60 & -75 \end{pmatrix}$$

$$\sim \begin{pmatrix} \boxed{1} & 2 & 3 & 4 & 5 \\ 0 & \boxed{-5} & -10 & -15 & -20 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 2 & 3 & 4 & 5 \\ 0 & \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} \boxed{1} & 0 & -1 & -2 & -3 \\ 0 & \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The fact that W has a basis of two elements is already clear after two steps, and the super-reduction was unnecessary. But we can now give three useful bases:

$$W = \mathscr{L}\left\{\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}, \begin{pmatrix}0\\1\\2\\3\\4\end{pmatrix}\right\} = \mathscr{L}\left\{\begin{pmatrix}-1\\0\\1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\1\\2\\3\\4\end{pmatrix}\right\} = \mathscr{L}\left\{\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}, \begin{pmatrix}6\\7\\8\\9\\10\end{pmatrix}\right\}.$$

L8.4 Dimension. First we reassure ourselves that any subspace of $\mathbb{R}^n$ has a basis.

Theorem. *Let V be a subspace of $\mathbb{R}^n$ which is not null. Then V has a basis consisting of at most n elements.*

Proof. By assumption, V contains a nonzero vector $\mathbf{u}_1$. Then

either $V = \mathscr{L}\{\mathbf{u}_1\}$ or we can choose $\mathbf{u}_2 \in V \setminus \mathscr{L}\{\mathbf{u}_1\}$.

In the latter case, $\mathbf{u}_2$ cannot be a multiple of $\mathbf{u}_1$, and

either $V = \mathscr{L}\{\mathbf{u}_1, \mathbf{u}_2\}$ or we can choose $\mathbf{u}_3 \in V \setminus \mathscr{L}\{\mathbf{u}_1, \mathbf{u}_2\}$.

In the latter case $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is LI since any linear relation could be used to express $\mathbf{u}_3$ as a LC of $\mathbf{u}_1, \mathbf{u}_2$, and the procedure can be continued. The set of elements $\mathbf{u}_1, \mathbf{u}_2, \ldots$ that we have selected at each stage is necessarily LI for the same reason. But we know that it is impossible to have more than n elements of $\mathbb{R}^n$ that are LI. So the process must stop. QED

Recall that if the rows of a matrix $A \in \mathbb{R}^{m,n}$ are LI then $r(A) = m$. A deeper fact we have seen is that if A, B are two matrices of the same size with $\text{Row}\,A = \text{Row}\,B$ then $A \sim B$.

Corollary. *Let V be a subspace of $\mathbb{R}^n$ that is not null. Any two bases of V have the same number of elements.*

Proof. Given two bases $\{\mathbf{a}_1, \ldots, \mathbf{a}_\ell\}$ and $\{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ with $\ell < m$, make them the rows of matrices A and B, adding null rows to A so that A, B both have size $m \times n$:

$$A = \begin{pmatrix} \leftarrow \mathbf{a}_1 \rightarrow \\ \cdots \\ \leftarrow \mathbf{a}_\ell \rightarrow \\ \hline \leftarrow \mathbf{0} \rightarrow \end{pmatrix}, \qquad B = \begin{pmatrix} \leftarrow \mathbf{b}_1 \rightarrow \\ \leftarrow \mathbf{b}_2 \rightarrow \\ \cdots \\ \leftarrow \mathbf{b}_m \rightarrow \end{pmatrix}.$$

The ranks of A, B are ℓ, m respectively. But since both sets of rows generate the same subspace V, we must have $\text{Row}\,A = \text{Row}\,B$. From the earlier fact $A \sim B$, and so $r(A) = r(B)$, a contradiction. QED

The number of elements in a basis is called the *dimension* of V, and we write it as $\dim V$. For a subspace of $\mathbb{R}^n$, we know that $\dim V \leqslant n$. Thus,

$$V = \mathscr{L}\{\underbrace{\mathbf{u}_1, \ldots, \mathbf{u}_k}_{LI}\}, \qquad k = \dim V.$$

If $V = \{\mathbf{0}\}$ then we set $\dim V = 0$, and (by convention) declare $\emptyset$ to be a basis.

Our final big result before turning to more geometrical applications is

Theorem. *For any matrix A of size $m \times n$,*

$$\dim(\text{Row}\,A) = r(A), \qquad \dim(\text{Ker}\,A) = n - r(A).$$

Proof. We may suppose that A is step-reduced as none of Row A, Ker A, $r(A)$ changes under ERO's. The nonzero rows of A then form a basis of Row A, whence the first equality. The second equality is then a restatement of (RC2), whereby the solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ depend on $n - r(A)$ free parameters. More precisely, a basis of the Ker A is given by the solutions of the form

$$\mathbf{x} = (a_1, \dots, a_{j-1}, -1, 0 \dots, 0)^T,$$

of which there is one for each unmarked column. QED

We have also stated that the marked columns of B (of which there are $r(B)$ in number) form a basis of Col B, though the latter *does* in general change with ERO's. But the columns of two row equivalent matrices satisfy the same linear relations, so the same columns will form a basis of Col A. Thus,

$$\dim(\text{Col } A) = r(A).$$

Corollary. $r(A) = r(A^{\top})$ *for any matrix* A*.*

This result is important as it means all the definitions and procedures that we carried out would have given the same results had we interchanged the roles of rows and columns. We shall return to study the column space of a matrix in Part II.

L8.5 Further exercises.

1. Find the dimensions of the following subspaces of $\mathbb{R}^5$:

$$U = \mathscr{L}\{(1,3,-2,2,3), (1,4,-3,4,2), (2,3,-1,-2,9)\},$$
$$V = \mathscr{L}\{(1,3,0,2,1), (1,5,-6,6,3), (2,5,3,2,1)\}.$$

Let W denote the subspace generated by all six row vectors. Find $\dim W = 5$.

2. Find the dimensions of the following subspaces of $\mathbb{R}^5$:

$$U = \{(x_1, x_2, x_3, x_4, x_5) : 2x_1 - x_2 - x_3 = 0,\ x_4 - 3x_5 = 0\},$$
$$V = \{(x_1, x_2, x_3, x_4, x_5) : 2x_1 - x_2 + x_3 + 4x_4 + 4x_5 = 0\}.$$

Let W denote the subspace consisting of vectors satisfying all three equations. Is it true that $\dim W = 5$?

3. Consider the following vectors of $\mathbb{R}^4$:

$$\mathbf{w}_1 = (1,0,1,0), \quad \mathbf{w}_2 = (2,h,2,h), \quad \mathbf{w}_3 = (1,1+h,1,2h).$$

Find the dimension of $\mathscr{L}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ as h varies.

4. Find a basis of $\mathbb{R}^4$ that contains both a basis for U and basis of V, where

$$U = \{(x,y,z,t) \in \mathbb{R}^4 : x - 2z = y = 0\},$$
$$V = \mathscr{L}\{(0,2,1,-1), (1,-2,1,1), (1,2,3,-1), (1,2,7,1)\}.$$

5. Explain carefully why the marked columns of a step-reduced matrix B form a basis of Col B.