Notes 7 – Subpaces of $\mathbb{R}^n$

We can understand the theory of matrices better using the concept of *subspace.*

L7.1 Closure. We continue to use $\mathbb{R}^n$ to denote either the set $\mathbb{R}^{1,n}$ of row vectors, or the set of column vectors $\mathbb{R}^{n,1}$. The definitions in this lecture apply equally to both cases, though at times it is best to specify one or the other.

Definition. *A* subspace *of $\mathbb{R}^n$ is a nonempty subset V that is 'closed' under addition and multiplication by a constant, meaning that these operations do not allow one to escape from the subset V (like a room with closed doors!).*

Thus, V is a sub*space* iff

(S1) $\mathbf{u},\mathbf{v}\in V \;\Rightarrow\; \mathbf{u}+\mathbf{v}\in V$,

(S2) $a\in\mathbb{R},\ \mathbf{v}\in V \;\Rightarrow\; a\mathbf{v}\in V$.

The word 'space' conveys the fact that, with these operations, the subset V acquires a structure of its own without the need to refer to $\mathbb{R}^n$. It is an immediate consequence that any subspace must contain the null vector. For if $\mathbf{v}\in V$ then

$$\mathbf{0} = \mathbf{v} + (-1)\mathbf{v} \in V.$$

Moreover, the singleton set $\{\mathbf{0}\}$ consisting of *only* the null vector is always a subspace. It is called the *null subspace* or *zero subspace* and any other subspace of $\mathbb{R}^n$ must have *infinitely many elements*. 𝔚𝔞𝔯𝔫𝔦𝔫𝔤: do not confuse the null subspace with the empty set $\emptyset$ that is *not* counted as a subspace.

At the other extreme is $\mathbb{R}^n$ itself. There is no doubt that this is a subspace, as conditions (S1) and (S2) are satisfied by default: the vectors $\mathbf{u}+\mathbf{v}$ and $a\mathbf{v}$ are certainly in $\mathbb{R}^n$ as they have nowhere else to go!

> **Example.** To test whether a subset of $\mathbb{R}^n$ is a sub*space,* check first that it contains $\mathbf{0}$. Be careful though; here are two subsets of the plane that both contain $\mathbf{0}$ but are not subspaces:
>
> (a) $A = \{(x,y)\in\mathbb{R}^2 : x \geqslant 0$ and $y \geqslant 0$, geometrically the first quadrant; it satisfies (S1) but not (S2).
>
> (b) $B = \{(x,y)\in\mathbb{R}^2 : xy = 0\}$, geometrically the union of the two axes; it satisfies (S2) but not (S1).

In practice, subspaces are constructed by taking linear combinations of vectors:

Lemma. *Any subset $\mathscr{L}\{\mathbf{u}_1,\dots,\mathbf{u}_k\}$ (with each $\mathbf{u}_i\in\mathbb{R}^n$) is a subspace.*

Proof. For simplicity, suppose that we have only two vectors $\mathbf{u}_1=\mathbf{u}$, $\mathbf{u}_2=\mathbf{v}$. Two arbitrary elements of $\mathscr{L}\{\mathbf{u},\mathbf{v}\}$ are then $a\mathbf{u}+b\mathbf{v}$, $c\mathbf{u}+d\mathbf{v}$, and their sum

$$(a\mathbf{u}+b\mathbf{v}) + (c\mathbf{u}+d\mathbf{v}) = (a+c)\mathbf{u} + (b+d)\mathbf{v}$$

obviously stays in $\mathscr{L}\{\mathbf{u},\mathbf{v}\}$. So does any multiple of $a\mathbf{u}+b\mathbf{v}$. QED

The converse of this result is valid, namely that *any* subspace of $\mathbb{R}^n$ can be expressed in the form $\mathscr{L}\{\mathbf{u}_1,\dots,\mathbf{u}_k\}$. To see this, one chooses a succession $\mathbf{u}_1,\mathbf{u}_2,\dots$ of vectors in V, preferably in such a way that each one is not a LC of the previous ones. We shall explain this better in the next lecture.

L7.2 Solution spaces. The set of solutions of a homogeneous system considered previously had the form $V = \mathscr{L}\{\mathbf{u},\mathbf{v}\}$, where $\mathbf{u},\mathbf{v}$ were two column vectors, and is therefore a subspace. But there is a more basic reason for this:

Proposition. *Given a matrix $A \in \mathbb{R}^{m,n}$, the set*

$$\{\mathbf{x} \in \mathbb{R}^{n,1} \;:\; A\mathbf{x} = \mathbf{0}\} \tag{1}$$

of solutions of the associated homogeneous linear system is always a subspace of $\mathbb{R}^{n,1}$.

Proof. This follows from the corresponding properties of matrix multiplication. If $\mathbf{x},\mathbf{y}$ are solutions then

$$A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0}+\mathbf{0} = \mathbf{0},$$

so $\mathbf{x}+\mathbf{y}$ is a solution too. Similarly,

$$A(a\mathbf{x}) = a(A\mathbf{x}) = a\mathbf{0} = \mathbf{0},$$

and $a\mathbf{x}$ is a solution for any $a \in \mathbb{R}$. QED

Definition. *The subspace (1) is called the* null *space or* kernel *of the matrix A, and denoted* Ker A.

> **Example.** Let W denote the set of vectors (x,y,z) that satisfy $x+y+z=0$. Since this is effectively a linear system (with $m=1$ and $n=3$), W *is* a subspace of $\mathbb{R}^3$. But we can easily express it as a LC by picking a couple of elements in it. Let $\mathbf{u} = (1,-1,0)$ and $\mathbf{v} = (0,1,-1)$. Both lie in W since their entries add up to 0. But we claim that any element (x,y,z) of W is a LC of $\mathbf{u}$ and $\mathbf{v}$. Indeed,
>
> $$(x,y,z) = (x,\ -x-z,\ z) = x\mathbf{u} - z\mathbf{v},$$
>
> as claimed. Thus $W = \mathscr{L}\{\mathbf{u},\mathbf{v}\}$.

$\mathfrak{Warning}$: The solution set is only a subspace when the system is *homogeneous.* For a inhomogeneous system $A\mathbf{x} = \mathbf{b}$, the solution set has the form

$$\{\mathbf{x}_0 + \mathbf{v} \;:\; \mathbf{v} \in \operatorname{Ker} A\}.$$

Here, $\mathbf{x}_0$ is any *particular* solution of the inhomogeneous equation $A\mathbf{x} = \mathbf{b}$; the difference of any two such solutions $\mathbf{x}_0, \mathbf{x}_1$ belongs to Ker A because $A(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}$.

L7.3 Subspaces defined by a matrix. Given a matrix A, two separate collections of vectors are staring us in the face:

the rows $\mathbf{r}_1,\cdots,\mathbf{r}_m \in \mathbb{R}^{1,n}$ of A, and
the columns $\mathbf{c}_1,\cdots\cdots,\mathbf{c}_n \in \mathbb{R}^{m,1}$ of A.

These give rise to two respective subspaces that complement the one already defined in (1).

Definition. *With the notation above,*
(i) *the* row space *of A, denoted* Row A, *is* $\mathscr{L}\{\mathbf{r}_1,\cdots,\mathbf{r}_m\} \subset \mathbb{R}^{1,n}$, *and*
(ii) *the* column space *of A, denoted* Col A, *is* $\mathscr{L}\{\mathbf{c}_1,\cdots\cdots,\mathbf{c}_n\} \subset \mathbb{R}^{m,1}$.
More informally, Row A *is a subsapce of $\mathbb{R}^n$, whereas* Col A *is a subspace of $\mathbb{R}^m$.*

Each row of A corresponds to an equation of the linear system with augmented matrix $(A\,|\,\mathbf{0})$. We already know that there are many ways to transform this system into an equivalent one with the same solutions. The next result formalizes the fact that it is the *row space* $\mathrm{Row}\,A$ (rather than the individual rows of A) that determines the solution space $\mathrm{Ker}\,A$.

Lemma. $\mathrm{Ker}\,A = \{\mathbf{x} \in \mathbb{R}^{m,1} \ : \ \mathbf{r}\mathbf{x} = 0$ *for all* $\mathbf{r} \in \mathrm{Row}\,A\}$.

Proof. Since

$$A\mathbf{x} = \begin{pmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \cdot \\ \mathbf{r}_m \cdot \mathbf{x} \end{pmatrix},$$

$\mathbf{x}$ belongs to $\mathrm{Ker}\,A$ iff $\mathbf{r}_i\mathbf{x} = 0$ for all i. This implies that $\mathbf{r}\mathbf{x} = 0$ for any $\mathbf{r} \in \mathrm{Row}\,A$ since such an $\mathbf{r}$ is a LC of the rows $\mathbf{r}_1, \cdots, \mathbf{r}_m$. Conversely, if $\mathbf{r}\mathbf{x} = 0$ for all $\mathbf{r} \in \mathrm{Row}\,A$ then certainly $\mathbf{r}_i\mathbf{x} = 0$ for all i, and so $A\mathbf{x} = 0$. QED

Recall the notion of row equivalence. It is easy to see that

$$A \sim B \quad \Rightarrow \quad \mathrm{Row}\,A = \mathrm{Row}\,B. \tag{2}$$

For if $A \sim B$, each row of B is obtained from A using ERO's, and $\mathrm{Row}\,B \subseteq \mathrm{Row}\,A$. But the process is reversible: $B \sim A$ and $\mathrm{Row}\,A \subseteq \mathrm{Row}\,B$. It follows from the Lemma that

$$A \sim B \quad \Rightarrow \quad \mathrm{Ker}\,A = \mathrm{Ker}\,B, \tag{3}$$

confirming something we already know: *if two matrices A, B are row equivalent then the associated homogeneous systems have the same solutions.*

We can complete these observations by the next result, which is easily memorized.

Theorem. *Let A, B be two matrices of the size $m \times n$. The following are equivalent:*

(i) *$A \sim B$, i.e. A and B are related by ERO's.*
(ii) $\mathrm{Row}\,A = \mathrm{Row}\,B$,
(iii) $\mathrm{Ker}\,A = \mathrm{Ker}\,B$.

This is especially relevant in the case in which B is a step-reduced matrix obtained by applying ERO's to A. Notice that the statement $A \sim B$ forces the matrices to have the same size – one could relax this requirement (and retain the Theorem's validity) by introducing a fourth ERO, that of deleting null rows.

𝔚𝔞𝔯𝔫𝔦𝔫𝔤. $A \sim B$ does not imply that $\mathrm{Col}\,A = \mathrm{Col}\,B$; to see this reduce the matrix $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Proof. Assume (ii). We already know that this implies (iii). Applying ERO's does not change the row space (statement (2)), so we may assume that A and B are step-reduced. Since the systems $A\mathbf{x} = \mathbf{0}$, $B\mathbf{x} = \mathbf{0}$ have the same solutions, we know that the markers of A and B occur in the same positions, and we may as well suppose that neither has a null row.

To deduce (i), one uses an 'exchange' technique that we shall describe by means of an example with $n = 5$. Let $\mathbf{a}_i$ be the rows of A, and $\mathbf{b}_i$ those of B; the idea is to slowly replace the former by the latter by the process illustrated:

$$A' = \begin{pmatrix} \leftarrow \mathbf{b}_1 \rightarrow \\ \leftarrow \mathbf{b}_2 \rightarrow \\ \hline \leftarrow \mathbf{a}_1 \rightarrow \\ \leftarrow \mathbf{a}_3 \rightarrow \\ \leftarrow \mathbf{a}_4 \rightarrow \end{pmatrix}, \qquad A'' = \begin{pmatrix} \leftarrow \mathbf{b}_1 \rightarrow \\ \leftarrow \mathbf{b}_2 \rightarrow \\ \leftarrow \mathbf{b}_3 \rightarrow \\ \hline \leftarrow \mathbf{a}_1 \rightarrow \\ \leftarrow \mathbf{a}_3 \rightarrow \end{pmatrix}$$

Suppose that we have already shown that A is row equivalent to A', in which $\mathbf{a}_2, \mathbf{a}_5$ have been replaced by $\mathbf{b}_1, \mathbf{b}_2$. Since

$$\mathbf{b}_3 \in \mathrm{Row}\, B = \mathrm{Row}\, A = \mathrm{Row}\, A'$$

(the last equality by (2)), $\mathbf{b}$ is a LC of the rows of A'. In this LC, one of $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4$ must figure with a nonzero coefficient, otherwise $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ would not be LI. If (say) $\mathbf{a}_4$ features then A'' can be obtained from A by a sequence of ERO's, and $A \sim A' \sim A''$. We can repeat the process and conclude at the end that $A \sim B$.

To see that (iii) implies (ii), we need the formula

$$\mathrm{Row}\, A = \{\mathbf{r} \in \mathbb{R}^{1,n} \ : \ \mathbf{r}\mathbf{x} = 0 \text{ for all } \mathbf{x} \in \mathrm{Ker}\, A\}$$

that is the counterpart of the Lemma. It is obvious that Row A is contained in the right-hand side, though the equality is best proved using the notion of dimension that will be discussed in the next lecture. QED

L7.4 Further exercises.

1. Use ERO's to show that the following subspaces of $\mathbb{R}^4$ coincide:

$$\mathscr{L}\{(1,2,-1,3),(2,4,1,-2),(3,6,3,-7)\}, \qquad \mathscr{L}\{(1,2,-4,11),(2,4,-5,14)\}.$$

2. Consider the following subspaces of $\mathbb{R}^4$:

$$U = \mathscr{L}\{(1,2,-1,3),(2,4,1,-2),(3,6,3,-7)\}, \qquad V = \mathscr{L}\{(1,2,-4,11),(2,4,0,14)\}.$$

Is it true that $U \subseteq V$ or $V \subseteq U$?

3. Let $W = \{(x,x,xy,y,y) : x,y \in \mathbb{R}\}$. Which of the following statements is true?
   (i) W is a subspace of $\mathbb{R}^5$,
   (ii) W is contained in a *proper* subspace U (so $W \subseteq U \neq \mathbb{R}^5$),
   (iii) W contains a subspace V that is not null (so $\mathbf{0} \neq V \subseteq W$).

4. Show that

$$\mathrm{Col}\, A = \{\mathbf{x}^\top \in \mathbb{R}^{m,1} \ : \ \mathbf{x} \in \mathrm{Row}(A^\top)\}.$$

This means that Col A is effectively the same as $\mathrm{Row}(A^\top)$. We could define a fourth subspace $\mathrm{Ker}(A^T) = \{\mathbf{x} \in \mathbb{R}^{m,1} \ : \ \mathbf{x}^\top A = \mathbf{0}\}$, but we have enough work to do studying Row A and Ker A for the time being!