Notes 6 – Solving a general system

Having introduced the *rank* of an arbitrary matrix, we are in a position to formulate the celebrated results of E. Rouché (1832–1910) and A. Capelli (1855–1910) concerning solutions of a (generally inhomogeneous) linear system of equations. *From now on, we denote the rank of a matrix M by $r(M)$.*

L6.1 The augmented matrix. Let us return to the inhomogeneous system in matrix form

$$A\mathbf{x} = \mathbf{b}, \qquad A \in \mathbb{R}^{m,n}, \quad \mathbf{x} \in \mathbb{R}^{n,1}, \quad \mathbf{b} \in \mathbb{R}^{m,1}.$$

We associate with this the so-called *augmented matrix*

$$(A \,|\, \mathbf{b}) = \left(\begin{array}{cccc|c} a_{11} & \cdot & \cdot & a_{1n} & b_1 \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ a_{m1} & \cdot & \cdot & a_{mn} & b_m \end{array}\right). \tag{1}$$

The vertical bar reminds us that the last column is special, but in applying row operations it should be ignored so that $\mathbf{b}$ is just treated as an extra column added to A. We solve the system by applying ERO's to (1) so that it becomes a step-reduced matrix

$$(A' \,|\, \mathbf{b}'). \tag{2}$$

𝔚𝔞𝔯𝔫𝔦𝔫𝔤: in doing this it is essential to apply each ERO to the whole row, including b_i; thus the last column will usually change unless it was already null.

Note that if (2) is step-reduced, so is the left-hand matrix A'. By definition then,

$$r(A) = r(A'), \qquad r(A \,|\, \mathbf{b}) = r(A' \,|\, \mathbf{b}'),$$

where we are relying on the previous Theorem. Furthermore,

$$r(A') \leqslant r(A' \,|\, \mathbf{b}') \leqslant r(A') + 1,$$

and we consider the two possibilities in turn. First, we record the

Lemma. *The m rows of a matrix $A \in \mathbb{R}^{m,n}$ are LI if and only if $r(A) = m$. In particular, since $r(A) \leqslant n$, no set of LI vectors in $\mathbb{R}^n$ has* more *than n elements.*

Proof (of 'only if'). Suppose that the rows of A are LI but that $r(A) < m$. In this case, when ERO's are performed to convert A into a step-reduced matrix A' the last row $\mathbf{r}_m$ of A' will be null. But (inverting the operations one by one) $\mathbf{r}_m$ is ultimately a nontrivial LC of the rows of A, contradiction. QED

Think of LI rows as 'incompressible': when the matrix is reduced they are not diminished in number.

Exercise. (i) The columns of a matrix A are LI iff the matrix equation $A\mathbf{x} = \mathbf{0}$ admits only the trivial solution $\mathbf{x} = \mathbf{0} \in \mathbb{R}^{m,1}$.

(ii) The rows of a matrix are LI iff the equation $\mathbf{x}A = \mathbf{0}$ only has the trivial solution $\mathbf{x} = \mathbf{0} \in \mathbb{R}^{1,n}$.

(iii) The rows of a matrix are LI iff the equation $A^\top \mathbf{x} = \mathbf{0}$ only has the trivial solution $\mathbf{x} = \mathbf{0} \in \mathbb{R}^{n,1}$.

L6.2 Inconsistent systems. Given a linear system, the student-friendly situation is that in which there are no solutions, as one does not have to waste time finding them!

Proposition. (RC1) *If* $r(A) < r(A\,|\,\mathbf{b})$*, the system has no solutions.*

This case can only occur if $r(A)$ is less than the number m of its rows, since otherwise both matrices will have rank m.

Proof. Let $r = r(A) < r(A\,|\,\mathbf{b})$. Then the first null row of A' is the $(r+1)$st and will be followed by $b_{r+1} \neq 0$ in the step-reduced matrix (2). This row represents a contradictory equation

$$0x_1 + \cdots + 0x_n = b_{r+1},$$

and the only way out is that the x_i do not exist.

L6.3 Counting parameters. The consistent case is characterized by the

Proposition. (RC2) *If* $r(A) = r(A\,|\,\mathbf{b})$*, there exist solutions depending upon* $n-r$ *parameters where* r *is the common rank. If* $n = r$ *there is a* unique *solution.*

Of course, if the system is homogeneous we must be in this case since A and $(A\,|\,\mathbf{0})$ only differ by a null column.

Proof. Each column of A and A' corresponds to a variable, so we can speak of 'marked' and 'unmarked' variables. It is easier (but not essential) to assume that B^+ is super-reduced, in which case its ith row has the form

$$(0\ \cdots\ 0\ \boxed{1}\ ?\ \ldots\ ?\,|\,c_i),$$

and represents an equation

$$\text{marked variable} + \text{LC of unmarked variables} = c_i.$$

It follows that we can assign the unmarked variables arbitrarily and solve uniquely for each of the marked variables in terms of them. QED

In the light of the procedure above, the unmarked variables are called *free variables,* and in the solution it is good practice to give them new names such as $s, t, u \cdots$ or $t_1, t_2, t_3 \ldots$. The conclusion is traditionally expressed by the statement

'If $r(A) = r(A\,|\,\mathbf{b}) = r$ then the linear system has ∞^{n-r} solutions'.

This is a useful way of recording the result that can be understood as follows. The actual number m of equations is irrelevant; what is important is the number of LI or *effective* equations, and this is the rank r. Each effective equation allows us to express one of the n variables in terms of the others, so we end up with $n - r$ free variables or parameters.

L6.4 Inversion by reduction.

Having introduced the augmented matrix, we can apply similar techniques to solve matrix equations of the type $AX = B$ where X and B are matrices rather than just column vectors. A special case is

$$AX = I_n, \qquad A, X \in \mathbb{R}^{n,n},$$

whose solution X (if it exists) is necessarily A^{-1}. As a consequence,

Proposition. *If $A \in \mathbb{R}^{n,n}$ is invertible then the unique super-reduced form of $(A\,|\,I_n)$ is $(I_n\,|\,A^{-1})$.*

Here is an example:

$$(A\,|\,I_5) = \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & 5 & 8 & 0 & 1 & 0 \\ 13 & 21 & 35 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & -3 & 1 & 0 \\ 13 & 21 & 35 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & -3 & 1 & 0 \\ 0 & 8 & 9 & -13 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & -1 & -4 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 9 & -2 \\ 0 & 0 & 1 & -1 & -4 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & 8 & -2 \\ 0 & 2 & 0 & -1 & 9 & -2 \\ 0 & 0 & 1 & -1 & -4 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & 8 & -2 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{9}{2} & -1 \\ 0 & 0 & 1 & -1 & -4 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{2} & \frac{7}{2} & -1 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{9}{2} & -1 \\ 0 & 0 & 1 & -1 & -4 & 1 \end{array}\right).$$

confirming the inverse found in L2. The matrices on the right act as a 'book-keeping' of the ERO's which there is no need for us to record separately.

The reason the method works is that each of the three types of ERO's is actually achieved by pre-multiplying A by a suitable invertible matrix E_i. For example, the first two are achieved by

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -13 & 0 & 1 \end{pmatrix}$$

respectively. By the time we have finished, the last line tells us that

$$E_7E_6E_5E_4E_3E_2E_1A = I_5 \quad \Rightarrow \quad A^{-1} = E_7E_6E_5E_4E_3E_2E_1I_5.$$

Corollary. *If $A \sim B$ there exists an invertible matrix $E \in \mathbb{R}^{m,m}$ such that $EA = B$.*

L6.5 Further exercises.

1. Consider the linear systems

$$\text{(i)} \begin{cases} -x_1 - 2x_2 + x_3 = 0 \\ x_1 + 3x_2 - x_3 = a, \end{cases} \qquad \text{(ii)} \begin{cases} x_1 - 2x_2 = 1 \\ 2x_1 + x_2 = 2 \\ x_1 - x_2 = b, \end{cases} \qquad \text{(iii)} \begin{cases} -x_1 - 2x_2 + x_3 - 2x_4 = 0 \\ 2x_1 + 3x_2 - 3x_3 + 3x_4 = 1 \\ x_2 + x_3 + x_4 = c. \end{cases}$$

Is it true that for every $a \in \mathbb{R}$, (i) has infinitely many solutions?
Is it true that (ii) never has a solution irrespective of the value of b?
Is it true that for all $c \in \mathbb{R}$, (iii) has a solution?

2. Given the system $\begin{cases} x_1 + x_2 + x_3 = k \\ x_1 - kx_2 + x_3 = -1 \\ -x_1 + kx_2 + x_3 = k, \end{cases}$ find all solutions in the case that $k = -1$.

Then discuss the existsence of solutions as k varies.

3. Find a relation between h_1, h_2, h_3 in order that

$$\begin{cases} x_1 - 2x_2 + x_3 + 2x_4 &= h_1 \\ x_1 + 3x_2 + x_3 - 3x_4 &= h_2 \\ 2x_1 + x_2 + 2x_3 - x_4 &= h_3 \end{cases}$$

has a solution. Find the general solution when $h_1 = -1$, $h_2 = 4$, $h_3 = 3$.

4. Use ERO's to reduce the matrix $A = \begin{pmatrix} 2 & \frac{5}{2} & 3 \\ 4 & 5 & a \\ b & b & b \end{pmatrix}$ with $a, b \in \mathbb{R}$. One of the following statements is false. Which?

(i) If $b = 0$ then $r(A) \leqslant 2$,
(ii) A is invertible if $a = b = 1$,
(iii) A is invertible if $a \neq 6$,
(iv) $r(A) \geqslant 1$ for any $a, b \in \mathbb{R}$.

5. Find values of $t \in \mathbb{R}$ for which each of the following matrices is not invertible:

$$\begin{pmatrix} 1 & -t \\ t & 4 \end{pmatrix}, \quad \begin{pmatrix} 3-t & -2 \\ -5 & -t \end{pmatrix}, \quad \begin{pmatrix} -t & -2 & -3 \\ 0 & 1-t & 1 \\ 1 & 2 & -t \end{pmatrix}, \quad \begin{pmatrix} -t & 3 & -3 & -6 \\ 0 & -t & 0 & 0 \\ 1 & 1 & -t & 0 \\ 1 & 0 & 0 & -t \end{pmatrix}.$$

6. Find the values of $\lambda \in \mathbb{R}$ for which $A = \begin{pmatrix} \lambda & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & \lambda \end{pmatrix}$ is invertible. Now set $\lambda = 1$, and solve the matrix equation $AX = B$ where $X \in \mathbb{R}^{3,2}$ and $B = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}$.

7. Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 8 \\ 2 \end{pmatrix}$, verify that the equation $AX = B$ has an infinite number of solutions and determine the number of free paraemters.

8. Given $A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$, which of the following equations admit at least one solution?

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad A^2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

9. Give the matrices

$$A = \begin{pmatrix} 2 & -3 & -2 & 1 \\ 4 & -6 & 1 & -2 \\ 6 & -9 & -1 & -1 \end{pmatrix}; \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}; \quad B_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, B_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

determine the solutions of the matrix equations $AX = B_1$, $AX = B_2$, $AX = B_3$.