Notes 4 – Row equivalence

For simplicity, we shall first study *homogeneous* systems of equations. The secret is to configure the rows of the coefficient matrix A so as to (more or less) read off the solutions.

L4.1 Row operations. Consider a homogeneous linear system in matrix form

$$A\mathbf{x} = 0, \quad \text{with} \quad A \in \mathbb{R}^{m,n}, \quad \mathbf{x} \in \mathbb{R}^{n,1}.$$

In this case, each equation is completely determined by the corresponding row of A, and we can encode the equations by the m rows

$$\left\{ \begin{array}{c} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \cdot \\ \mathbf{r}_m \end{array} \right.$$

of A. In this notation, with $m = 4$, the scheme

$$\left\{ \begin{array}{c} \mathbf{r}_1 \\ \mathbf{r}_2 - 2\mathbf{r}_1 \\ \mathbf{r}_3 \\ 3\mathbf{r}_4 \end{array} \right.$$

represents an equivalent system of equations; we have merely subtracted twice the first from the second and multiplied the last by 3. These changes will not affect the values of any solution $(x_1, \ldots, x_n)$. We are also at liberty to change the order in which we list the equations.

Our aim is to use such changes to simplify the system.

Definition. *Let A be a matrix of size $m \times n$. An* elementary row operation (ERO) *is one of the following ways in which a new matrix of the same size is formed from A:*

(i) *add to a given row a multiple of a* different *row,*
(ii) *multiply a given row by a* nonzero *constant,*
(iii) *swap or interchange two rows.*

In symbols, we can denote the operations that we have just described by

(i) $\mathbf{r}_i \mapsto \mathbf{r}_i + a\mathbf{r}_j, \qquad i \neq j,$
(ii) $\mathbf{r}_i \mapsto c\mathbf{r}_i, \qquad c \neq 0,$
(iii) $\mathbf{r}_i \leftrightarrow \mathbf{r}_j.$

In practice, it is often convenient to take a to be negative; in particular (i) includes the act of subtracting one row from another: $\mathbf{r}_i \mapsto \mathbf{r}_i - \mathbf{r}_j$ (but it is essential that $i \neq j$ otherwise we would effectively have eliminated one of the equations).

L4.2 Solving a homogeneous system. Let us show how ERO's can be used to solve the linear system

$$\left\{ \begin{array}{rcl} x_1 + x_2 + 2x_3 + 3x_4 &=& 0 \\ 5x_1 + 8x_2 + 13x_3 + 21x_4 &=& 0 \\ 34x_1 + 55x_2 + 89x_3 + 144x_4 &=& 0 \end{array} \right. \tag{1}$$

written before. (The choice of *Mole* coefficients will keep the arithmetic manageable.)

We shall apply ERO's to convert A into a matrix that is roughly triangular, and then solve the resulting system.

$$\begin{array}{lcl} & A = & \begin{pmatrix} 1 & 1 & 2 & 3 \\ 5 & 8 & 13 & 21 \\ 34 & 55 & 89 & 144 \end{pmatrix} \\ \mathbf{r}_2 - 5\mathbf{r}_1 & \sim & \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 3 & 3 & 6 \\ 34 & 55 & 89 & 144 \end{pmatrix} \\ \tfrac{1}{3}\mathbf{r}_2 & \sim & \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 34 & 55 & 89 & 144 \end{pmatrix} \\ \mathbf{r}_3 - 34\mathbf{r}_1 & \sim & \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 21 & 21 & 42 \end{pmatrix} \\ \tfrac{1}{21}\mathbf{r}_3 & \sim & \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{pmatrix} \\ \mathbf{r}_3 - \mathbf{r}_2 & \sim & \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

On the left, we jot down (in abbreviated form) the operations used. It is not essential to do this, provided the operations are carried out one at a time; errors occur when one tries to be too ambitious! It follows from the last matrix that (1) has the same solutions as the system

$$\left\{ \begin{array}{rcl} x_1 + x_2 + 2x_3 + 3x_4 & = & 0 \\ x_2 + x_3 + 2x_4 & = & 0 \end{array} \right.$$

But one can see at a glance how to solve this; we can assign any values to x_3 and x_4 which will then determine x_2 (from the second equation) and then x_1 (from the first). Suppose that we set $x_3 = s$ and $x_4 = t$ (it is a good idea to use different letters to indicate free variables); then

$$x_2 = -s - 2t \quad \Rightarrow \quad x_1 = -(-s - 2t) - 2s - 3t = -s - t. \tag{2}$$

The general solution is

$$(x_1, x_2, x_3, x_4) = (-s - t,\ -s - 2t,\ s,\ t),$$

or in column form

$$\mathbf{x} = \begin{pmatrix} -s - t \\ -s - 2t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix},$$

where s, t are arbitrary. The *set* of solutions is therefore

$$\mathscr{L}\{\mathbf{u}, \mathbf{v}\}, \quad \text{where} \quad \mathbf{u} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$$

We shall see that the solution set of any homogeneous system is always a linear combination of this type.

Exercise. Compute $\mathbf{u} - \mathbf{v}$, and explain why this is also a solution.

L4.3 Equivalence relations. Applying ERO's produces a natural relation on the set of matrices of any fixed size.

Definition. *A relationship* $\sim$ *between elements of a set is called an* equivalence relation *if*

(E1) $A \sim A$ *is always true,*

(E2) $A \sim B$ *always implies* $B \sim A$*,*

(E3) $A \sim B$ *and* $B \sim C$ *always implies* $A \sim C$*.*

Observe that these three conditions are satisfied by equality $=$. On the set of real numbers, 'having the same absolute value' is an equivalence relation, but $\leqslant$ is not. But we are more interested in sets of matrices.

Definition. *From now on, we write* $A \sim B$ *to mean that* B *is a matrix obtained by applying one or more ERO's in succession to* A*.*

Proposition. *This does define an equivalence relation, and if* $A \sim B$ *we say that* A *and* B *are* row equivalent.

Proof. By definition $A \sim B$ and $B \sim C$ imply that $A \sim C$. Obviously $A \sim A$ (take (i) with $a = 0$ or (ii) with $c = 1$, both of which are permitted).

The condition (E2) is less obvious. But each of the three operations is invertible; it can be 'undone' by the same type of operation. For example, $\mathbf{r}_1 \mapsto \mathbf{r}_1 - a\mathbf{r}_2$ by $\mathbf{r}_1 \mapsto \mathbf{r}_1 + a\mathbf{r}_2$. So if $A \sim B$ then we can undo each ERO in the succession one at a time, and $B \sim A$. QED

Example. If B is a matrix which has the same rows as A but in a different order, then $A \sim B$. This is because any permutation of the rows can be obtained by a succession of transpositions, i.e. ERO's of type (iii). For example, let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \end{pmatrix}.$$

Then

B is obtained from A by $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$; C is obtained from B by $\mathbf{r}_2 \leftrightarrow \mathbf{r}_3$,

so $A \sim C$ even though it is not possible to pass from A to C by a single ERO. Of course, it is also true that $A \sim B$, $B \sim C$, and $C \sim A$.

Exercise. In this example, can C be obtained from A by a succession of ERO's of type (i)?

The theory of this lecture can be summarized by the

Proposition. *Suppose that* $A \sim B$*. Then the column vector* $\mathbf{x}$ *is a solution of* $A\mathbf{x} = \mathbf{0}$ *iff it is a solution of* $B\mathbf{x} = \mathbf{0}$*.*

In other words, the homogeneous linear systems associated to row equivalent matrices have the same solutions. This follows because each individual ERO leaves unchanged the solution space.

L4.4 Further exercises.

1. Use ERO's to find the general solution of the linear system

$$\begin{cases} 2x - 2y + z + 4t &= 0 \\ x - y - 4z + 2t &= 0 \\ -x + y + 3z - 2t &= 0 \\ 3x - 3y + z + 6t &= 0. \end{cases}$$

2. Show that the matrix

$$A = \begin{pmatrix} 1 & -2 & 1 & 2 \\ 1 & 3 & 1 & -3 \\ 1 & 8 & 1 & -8 \end{pmatrix}$$

is row equivalent to one with one row null.

3. Recall that a square matrix $P \in \mathbb{R}^{n,n}$ is called *invertible* if there exists a matrix P^{-1} of the same size such that $P^{-1}P = I_n = PP^{-1}$. Two matrices A, B are said to be *similar* if there exists an invertible matrix P such that $P^{-1}AP = B$. Prove that being similar is an equivalence relation on the set $\mathbb{R}^{n,n}$. (Hint: you will need the fact that $(P^{-1})^{-1} = P$.)