Notes 24 – Conics and Quadrics

In the last lecture, we discussed conics defined by setting a quadratic form equal to a constant. After defining a general conic, we list the eight different types and investigate when a conic has a centre of symmetry. We then move on to discuss quadrics from a similar point of view.

L24.1 Definition of a conic. A conic $\mathscr{C}$ is the set of points (x, y) in $\mathbb{R}^2$ determined by an equation of the form

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0, \tag{1}$$

where $A, \dots, F$ are real constants, and A, B, C are not all zero so that the left-hand side is a polynomial of degree 2 in the two variables x, y. (The 2's are for convenience later.)

Often we shall refer to an equation like (1) as a conic, though strictly speaking the latter is a set of points. The conics we discussed before were those for which $D = E = 0$. In this case, by diagonalizing the symmetric matrix

$$S = \begin{pmatrix} A & B \\ B & C \end{pmatrix}, \tag{2}$$

we can always rotate the coordinate system so that the equation of $\mathscr{C}$ becomes

$$\lambda_1 X^2 + \lambda_2 Y^2 = \mu, \qquad \mu = -f.$$

It follows that $\mathscr{C}$ is one of

(i) an ellipse (if $\lambda_1, \lambda_2, \mu$ all have the same sign); a circle is the special case in which $\lambda_1 = \lambda_2$;

(ii) a hyperbola (if λ_1, λ_2 have opposite signs and $\mu \neq 0$); the corresponding special case $\lambda_1 = -\lambda_2$ gives rise to a *rectangular hyperbola* whose asymptotes are perpendicular;

(iii) two straight lines interecting in one point (if λ_1, λ_2 have opposite signs but $\mu = 0$);

(iv) two parallel lines (if one of λ_1, λ_2 is zero and the other has the same sign as μ);

(v) a single line (if one of λ_1, λ_2 is zero and $\mu = 0$, for then the equation actually defines two coincident lines, though only one is visible to the naked eye);

(vi) a point (if λ_1, λ_2 have the same sign but $\mu = 0$);

(vii) in all other cases, the set of points satisfying (1) is empty.

Allowing D, E to be nonzero produces only one other type, namely

(viii) a parabola (such as $x^2 + y = 0$, or less obviously $4x^2 + 6xy + 9y^2 + x = 0$).

Given the general equation (1), we can try first to eliminate the term $2Dx + 2Ey$ of degree 1 by a change of coordinates of type

$$\begin{cases} x = X + u, \\ y = Y + v. \end{cases} \tag{3}$$

This corresponds to a translation in which the new system OXY has its origin O at the old point $(x, y) = (u, v)$. Substituting (3) into (1), we see that the new term of degree 1 is

$$2AuX + 2B(Xv + uY) + 2CvY + 2DX + 2EY = 2(Au + Bv + D)X + 2(Bu + Cv + E)Y.$$

To eliminate all this, we need to solve the linear system with unknowns u, v and matrix

$$\left(\begin{array}{cc|c} A & B & -D \\ B & C & -E \end{array}\right). \tag{4}$$

Since the left-hand side of this matrix is twice (2), a solution may not be possible if $\det S = 0$.

Definition. *The conic $\mathscr{C}$ is* central *if there is a translation (3) that converts its equation into the form $A'X^2 + 2B'XY + C'Y^2 + F' = 0$. In this case, $(X, Y) \in \mathscr{C} \Leftrightarrow (-X, -Y) \in \mathscr{C}$, and the centre of symmetry is the point $(X, Y) = (0, 0)$ or $(x, y) = (u, v)$.*

From the analysis above, we know that there is only one case in which (4) is incompatible and $\mathscr{C}$ is not central, namely (viii).

Corollary. *The conic (1) can only be a parabola if $B^2 = 4AC$.*

> **Example.** Given the conic $x^2 + 4y^2 - 6x + 8y = 3$, we can locate a centre by completing the squares:
> $$(x-3)^2 - 9 + 4(y+1)^2 - 4 = 3.$$
> Thus $u = 3$, $v = -1$, and the equation becomes
> $$X^2 + 4Y^2 = 16, \quad \text{or} \quad \frac{X^2}{4^2} + \frac{Y^2}{2^2} = 1,$$
> which is an ellipse with width twice its height. In general, if the orginal equation has a term in xy, one needs to find the centre by solving (4).

L24.2 Central quadrics. One can carry out a parallel discussion in space by adding a third variable.

Definition. *A* quadric *$\mathscr{Q}$ is the locus of points (x, y, z) in $\mathbb{R}^3$ satisfying an equation of the form*

$$Ax^2 + By^2 + Cz^2 + 2Dyz + 2Ezx + 2Fxy + 2Gx + 2Hy + 2Iz + J = 0. \tag{5}$$

The word 'quadric' implies that (5) has order 2, so not all of A, B, C, D, E, F are zero.

Just as we did for conics in L23.1, one can list all the different types of quadrics; whilst there were 8 types of conics there are 15 types of quadrics. However, we shall only consider the more interesting cases in this course.

Let us start with an obvious example. The equation

$$x^2 + y^2 + z^2 = r^2$$

fits the definition (with $A = B = C = 1$, $J = -r^2$, and all other coefficients zero). It is of course a *sphere* of radius r with centre the origin. Indeed if $\mathbf{v} = (x, y, z)^T$, then the equation becomes $|\mathbf{v}|^2 = r^2$ or $|\mathbf{v}| = r$, and asserts that the distance of (x, y, z) from the origin is r (see L9.1).

In the light of the discussion of ellipses, it should now come as no surprise that the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

represents an *ellipsoid* that fits snugly into a box centred at the orgin of dimension $2a \times 2b \times 2c$.

Definition. *A* central quadric *is the locus of points* (x, y, z) *satisfying an equation*

$$Ax^2 + By^2 + Cz^2 + 2Dyz + 2Ezx + 2Fxy + J = 0, \tag{6}$$

or equivalently

$$(x\ y\ z)\begin{pmatrix} A & F & E \\ F & B & D \\ E & D & C \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -J.$$

The 3×3 matrix here is symmetric, and we can rewrite the equation as

$$\mathbf{v}^{\top} S\, \mathbf{v} = -J, \qquad \text{where} \qquad \mu = -J.$$

We know from L21.1 that there exists a 3×3 orthogonal matrix P so that $P^{-1}SP = P^{\top}SP$ is diagonal. We may also suppose that $\det P = 1$ (for if not, $\det P = -1$ and we merely replace P by $-P$ and note that $\det(-P) = 1$). It follows from remarks in L21.3 that P represents a rotation; thus we have the

Theorem. *Given a central quadric (6), it is possible to rotate the coordinate system about the origin in space so that in the new system the equation becomes*

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 = \mu. \tag{7}$$

The numbers $\lambda_1, \lambda_1, \lambda_3$ *are (in no particular order) the eigenvalues of* S.

Here are some examples of central quadrics in which the eigenvalues are all nonzero:
(i) an ellipsoid (if $\lambda_1, \lambda_2, \lambda_3, \mu$ all have the same sign);
(ii) a *hyperboloid of one sheet* (if for example $\lambda_1, \lambda_2, \mu$ are positive and $\lambda_3 < 0$);
(iii) a *hyperboloid of two sheets* (if for example λ_1, λ_2 are negative and λ_3, μ are positive),
(iv) a *cone* (if not all $\lambda_1, \lambda_2, \lambda_3$ have the same sign and if $\mu = 0$).
In the last case, the cone is *circular* if two of the eigenvalues are equal, otherwise it is called *elliptic*. We shall explain this case further in the next lecture.

L24.3 Paraboloids. In some ways the simplest equation in x, y, z of second order is

$$z = xy.$$

This is the equation (5) of a quadric for which all the coefficients are zero except $F = -I$. If we perform a rotation of $\pi/4$ of the xy plane corresponding to the matrix

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

then we can replace x by $\frac{1}{\sqrt{2}}(X - Y)$ and y by $\frac{1}{\sqrt{2}}(X + Y)$, and leave $z = Z$ alone. Our quadric $\mathscr{Q}$ becomes

$$z = \tfrac{1}{2}(X - Y)(X + Y), \qquad \text{or} \qquad 2Z = X^2 - Y^2.$$

This is an example of a *hyperbolic paraboloid* that resembles a 'saddle' for a horse, or (on a bigger scale) a 'mountain pass'. Any plane $X = c$ or $Y = c$ intersects $\mathscr{Q}$ in a parabola, whereas a plane $Z = c$ intersects $\mathscr{Q}$ in a hyperbola or (if $c = 0$) a pair of lines.

Paraboloids are quadrics that cannot be put into the form (6) or (7), and therefore possess no central point of symmetry. The standard form of a paraboloid is the equation

$$Z = aX^2 + bY^2.$$

If a, b have opposite signs, it is again a hyperbolic paraboloid. If a, b have the same sign, the quadric is easier to draw and is called an *elliptic hyperboloid* (*circular* if $a = b$). Its intersection with the plane $Z = a$ is an ellipse (circle).

L24.4 Further exercises.

1. For each of the following conics, find the centre (u, v) and the equations that results by setting $x = X + u$, $y = Y + v$: (i) $x^2 + y^2 + x = 3$, (ii) $3y^2 - 4xy - 4x = 0$, (ii) $3x^2 - xy + 2y = 9$.

2. Find the centre (u, v) of the conic $\mathscr{C} : x^2 + xy + y^2 - 2x - y = 0$, and a symmetric matrix S so that the equation becomes $(X, Y)S\begin{pmatrix} X \\ Y \end{pmatrix} = 1$ with $x = X + u$, $y = Y + v$. Diagonalize S, and sketch $\mathscr{C}$ relative to the original axes (x, y).

3. Let $\mathscr{S}$ be the sphere with centre $(3, 1, 1)$ passing through $(3, 4, 5)$. Find the radius of $\mathscr{S}$, and write down its equation.

4. Let π be the plane $x - 2y + 2z = 0$ and let O denote the origin $(0, 0, 0)$. Find
   (i) the line ℓ orthogonal to π that passes through O,
   (ii) the point P on ℓ a distance 6 from O with $z > 0$;
   (iii) a sphere $\mathscr{S}$ of radius 6 tangent to π at O.

5. Match up, in the correct order, the quadrics

$$x^2 = 3y^2 + z^2 + 1, \qquad z^2 = xy, \qquad x^2 + 2y^2 - z^2 = 1, \qquad -x^2 - y^2 + 2x + 1 = 0$$

with (i) a hyperboloid of 1 sheet, (ii) a hyperboloid of 2 sheets, (iii) a cone, (iv) a cylinder.

6. Show that the line ℓ with parametric equation $(x, y, z) = (1, -t, t)$ is contained in the quadric $\mathscr{Q} : x^2 + y^2 - z^2 = 1$. Draw $\mathscr{Q}$ and ℓ in the same coordinate system. Find a second line ℓ' that lies in $\mathscr{Q}$.

7. Decide which of the following equations describes the circular cone that is obtained when one rotates the line $\{(x, y, z) : x = 0,\ z = 2y\}$ around the z-axis:

$$x^2 + 4y^2 = z^2, \qquad 4x^2 + 4y^2 - z^2 = 0, \qquad 2(x^2 + y^2) - z^2 = 0, \qquad z = 4x^2 + 4y^2.$$

8. The quadrics $\mathscr{Q}_1 : z = x^2 + y^2$ and $\mathscr{Q}_2 : z = x^2 - y^2$ are both examples of paraboloids. Write down the equations of planes $\pi_1, \pi_2, \pi_3, \pi_4$ parallel to the coordinate planes but such that
   (i) $\mathscr{Q}_1 \cap \pi_1$ is a parabola, (ii) $\mathscr{Q}_1 \cap \pi_2$ is a circle,
   (iii) $\mathscr{Q}_2 \cap \pi_3$ is a hyperbola, (iv) $\mathscr{Q}_2 \cap \pi_4$ is a pair of lines.