Notes 23 – Quadratic forms and conics

After some general definitions, we shall study quadratic forms in two variables x, y. Such a form corresponds to a 2×2 symmetric matrix whose diagonalization enables us to simplify the quadratic form. The resulting equations typically describe ellipses or hyperbolas, although degenerate cases are the subject of some of the exercises.

L23.1 Homogeneous polynomials. We already know that any linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ is effectively multiplication by an $m \times n$ matrix. In particular, setting $m = 1$, a linear mapping $\ell\colon \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\ell(\mathbf{v}) = (a_1 \cdots a_n)\mathbf{v} = a_1x_1 + \cdots + a_nx_n, \quad \text{where} \quad \mathbf{v} = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}.$$

The sum $a_1x_1 + \cdots + a_nx_n$ is called a *linear form* in the n variables $x_1, \ldots, x_n$; each term in this sum has degree exactly 1 (no constants are allowed on their own as these would have degree 0). Such a sum is also called a *homogeneous polynomial of degree 1* in $x_1, \ldots, x_n$.

A *quadratic form* in $x_1, \cdots, x_n$ is a linear combination of terms of degree exactly 2, such as ${x_1}^2$, x_1x_2 and so on. We can construct a quadratic form from a square matrix by setting

$$q(\mathbf{v}) = \mathbf{v}^\top A\mathbf{v} = (x_1, \ldots, x_n)\begin{pmatrix} a_{11} & a_{12} & \cdots & \\ a_{21} & a_{22} & \cdots & \\ \cdot & \cdot & \cdots & \\ & & & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}.$$

Using a calculation (best done in the 3×3 case), or the summation formula for matrix multiplication, we obtain

$$q(\mathbf{v}) = a_{11}{x_1}^2 + a_{12}x_1x_2 + \cdots + a_{21}x_2x_1 + \cdots + a_{nn}{x_{nn}}^2 = \sum_{i,j=1}^{n} a_{ij}x_ix_j. \tag{1}$$

This function is also called a *homogeneous polynomial of degree 2* since the total power of each term equals exactly 2.

Since the coefficient of x_ix_j in (1) equals $a_{ij} + a_{ji}$ if $i \neq j$, we may as well suppose that A is a *symmetric* matrix. With this assumption, we can recover A from the quadratic form:

Example. Given the quadratic form

$$q(x, y, z) = 2x^2 + y^2 + 5z^2 + 6xy + 2yz,$$

we need to half the 'mixed' coefficients to find the off-diagonal entries of the associated symmetric matrix

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 1 & 1 \\ 0 & 1 & 5 \end{pmatrix},$$

so that

$$q(x, y, z) = (x, y, z)A\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

L23.2 A 2×2 example revisited. Recall the matrix

$$A=\begin{pmatrix}5&2\\2&8\end{pmatrix}$$

studied in L21.1. It is associated to the quadratic form

$$q(x,y)=5x^2+4xy+8y^2.$$

and we now pose the problem: *describe the set of points (x,y) in $\mathbb{R}^2$ such that $q(x,y)=1$.* We shall answer this question by diagonalizing A.

The eigenvalues of A are 9 and 4. This time, we take the smallest first: $\lambda_1=4$ and $\lambda_2=9$. A first eigenvector lies in $\mathrm{Ker}(A-4I)=\mathrm{Ker}\,(1\ \ 2)$, and a unit one is $\frac{1}{\sqrt5}\begin{pmatrix}2\\-1\end{pmatrix}$. It follows that A is diagonalized by means of the orthogonal matrix

$$R_\theta=\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}=\frac{1}{\sqrt5}\begin{pmatrix}2&1\\-1&2\end{pmatrix};$$

there is no need to compute separately the second eigenvector. Thus,

$$R_{-\theta}\,A\,R_\theta=D=\begin{pmatrix}4&0\\0&9\end{pmatrix},\tag{2}$$

since $(R_\theta)^\top=R_{-\theta}=(R_\theta)^{-1}$.

We now define new coordinates X,Y by setting

$$\begin{pmatrix}X\\Y\end{pmatrix}=R_{-\theta}\begin{pmatrix}x\\y\end{pmatrix},\qquad\text{or}\qquad\left\{\begin{array}{l}X=\ x\cos\theta+y\sin\theta\\Y=-x\sin\theta+y\cos\theta,\end{array}\right.$$

For practical purposes, it is however best to express the *old* coordinates in terms of the *new* ones:

$$\begin{pmatrix}x\\y\end{pmatrix}=R_\theta\begin{pmatrix}X\\Y\end{pmatrix},\qquad\text{or}\qquad\left\{\begin{array}{l}x=X\cos\theta-Y\sin\theta\\y=X\sin\theta+Y\cos\theta,\end{array}\right.\tag{3}$$

For this enables us to *substitute* new for old, and the coefficients on the right of (3) are the entries of R_θ placed in the correct position. The linear mapping associated to R_θ is a rotation through an angle of θ, where

$$\cos\theta=\tfrac{2}{\sqrt5},\quad \sin\theta=-\tfrac{1}{\sqrt5},$$

which implies that θ is about -25.6°.

The transpose of the left-hand equation in (3) is $(x\ \ y)=(X\ \ Y)R_{-\theta}$. It follows from (2) that

$$(x\ \ y)A\begin{pmatrix}x\\y\end{pmatrix}=(X\ \ Y)R_{-\theta}\,A\,R_\theta\begin{pmatrix}X\\Y\end{pmatrix}=(X\ Y)\begin{pmatrix}4&0\\0&9\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix}.$$

Hence

$$5x^2+4xy+8y^2=4X^2+9Y^2,$$

and our equation becomes

$$4X^2+9Y^2=1\quad\text{or}\quad\frac{X^2}{(\frac12)^2}+\frac{Y^2}{(\frac13)^2}=1,\tag{4}$$

which is the equation of an ellipse.

L23.3 Drawing conics. To plot the ellipse

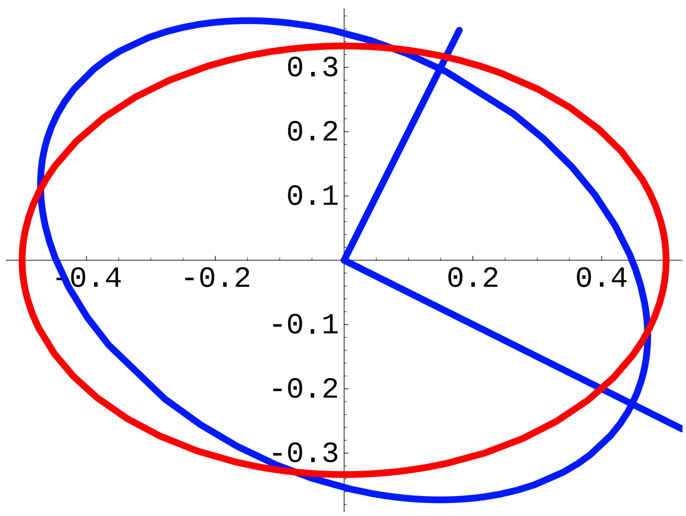
$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

centred at the origin, we construct a box of 'semi-width' a and 'semi-height' b, and fit the ellipse tightly into it. In the example (4), we have $a = \frac{1}{2}$ and $b = \frac{1}{3}$, and the result is shown below in red.

The clockwise rotation R_θ transforms a point $\begin{pmatrix} X \\ Y \end{pmatrix}$ on the red conic to a point $\begin{pmatrix} x \\ y \end{pmatrix}$ of the blue one. It follows that the blue conic represents the locus of points (x, y) satisfying the original equation

$$5x^2 + 4xy + 8y^2 = 1,$$

with the black axes now representing the old coordinates x, y.



A hyperbola

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$$

can be drawn in a similar way, by first constructing the box with centre the origin and size $2a \times 2b$. The diagonal lines form the *asymptotes* of the hyperbola, and taken together they have equation

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 0, \quad \text{or} \quad Y = \pm\frac{b}{a}X.$$

The two branches of the hyperbola itself are now easily drawn, and pass through the points $(-a, 0)$ and $(0, a)$ on the horizontal axis.

Example. Let c be a constant. The conic $4xy + 3y^2 = c$ is associated to the symmetric matrix

$$S = \begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}.$$

Since $\det S < 0$ the eigenvalues have opposite signs (they are 4 and -1) and the conic is a hyperbola provided $c \neq 0$. The matrix S is diagonalized by

$$P = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix},$$

and this corresponds to a rotation by $\theta = +25.6^{\circ}$.

L23.4 Further exercises.

1. Describe and then draw the conics
   (i) $4x^2 + y^2 = 4$, (ii) $4x^2 - y^2 = 0$, (iii) $4x^2 - y^2 = 4$.

2. Say whether there exists a column vector $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ such that $\mathbf{v}^{\top} A \mathbf{v} < 0$ when A is each of the following:
   (i) $\begin{pmatrix} 4 & -3 \\ -3 & 4 \end{pmatrix}$, (ii) $\begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}$, (iii) $\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$.

3. Write down the symmetric matrix $S \in \mathbb{R}^{2,2}$ associated to each of following quadratic forms $q(x,y)$, and find the signs of the eigenvalues of S:

$$x^2+4xy+3y^2, \qquad x^2-2xy+2y^2, \qquad x^2+4xy+4y^2, \qquad xy+y^2.$$

Determine the type of the conics $q(x,y)=1$, $q(x,y)=0$, $q(x,y)=-1$ in each case.

4. Factorize the polynomials $t^2 + 3t + 2$ and $t^2 + 4t + 4$. Use these results to describe the conics $x^2 + 3xy + 2y^2 = 0$ and $x^2 + 4xy + 4y^2 = 0$.

5. Reduce the conics
   (i) $x^2 + 4xy - 2y^2 = -1$, (ii) $x^2 + 3xy + 5y^2 = 10$
into canonical form $\lambda_1 X^2 + \lambda_2 Y^2 = 1$ and classify them.

6. (a) Which of the following equations could represent the hyperbola $X^2 - 3Y^2 = 1$ after a suitable rotation: (i) $x^2 - xy + y^2 = 1$, (ii) $x^2 + xy = 1$, (iii) $2x^2 - 2xy + y^2 = 1$?
(b) Find the values of a for which the conic $(a+3)x^2 + 4xy + ay^2 = 1$ is an ellipse.

7. Let $\mathscr{C}$ be the ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1,$$

and consider the points $F = (-3,0)$, $F' = (3,0)$. If $P = (x,y)$ is a point of $\mathscr{C}$, prove that the distance $|PF|$ satisfies $|PF|^2 = (5 + \frac{3}{5}x)^2$ and also that $|PF'|^2 = (5 - \frac{3}{5}x)^2$. Deduce that $|PF| + |PF'| = 10$, regardless of where P is on the ellipse.

8. Describe the conics defined by the following curves with parameter t:
   (i) $(t,\ t+t^2)$, (ii) $(2\cos t,\ \sin t)$, (iii) $(t+\frac{1}{t},\ t-\frac{1}{t})$.
In each case, plot that part of the conic with $0 < t < 1$.