Notes 21 – Diagonalizability

For an important class of square matrices (or linear transformations of a finite-dimensional vector space), it is possible to choose a *basis* of eigenvectors. We have already seen that this is possible if the characteristic polynomial of $A \in \mathbb{R}^{n,n}$ has distinct real roots. We shall now explain the significance of, and give a more general criterion for, the existence of a basis of eigenvectors. We include a well-known application to the theory of Fibonacci numbers.

L21.1 An example in detail. The following matrix featured in the first exercise of L20.4:

$$A = \begin{pmatrix} 5 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

One's first observation upon setting eyes on this matrix is that $A - I$ has a row of 0's; it follows that 1 is an eigenvalue of A. Given that

$$\det A = 5.1 - (-3).1 = 8, \qquad \operatorname{tr} A = 5{+}1{+}1 = 7,$$

we may choose the roots $\lambda_1, \lambda_2, \lambda_3$ of $p(x) = \det(A - xI)$ so that

$$\lambda_1 = 1, \qquad 1.\lambda_2\lambda_3 = 8 \qquad 1 + \lambda_2 + \lambda_3 = 7,$$

and $\lambda_2 = 2$ and $\lambda_4 = 4$. If required to do so, one can verify directly that

$$p(x) = -(x-1)(x-2)(x-4).$$

Eigenvectors can be found by picking particular solutions of the corresponding linear system:

$$A - I = \begin{pmatrix} 4 & 3 & -3 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = \begin{pmatrix} -6 \\ 3 \\ -5 \end{pmatrix}$$

$$A - 2I = \begin{pmatrix} 3 & 3 & -3 \\ 0 & -1 & 0 \\ 1 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$A - 4I = \begin{pmatrix} 1 & 3 & -3 \\ 0 & -3 & 0 \\ 1 & 2 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.$$

Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ denote the linear transformation defined by $f(\mathbf{v}) = A\mathbf{v}$. Instead of using the canonical basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of $\mathbb{R}^3$, we are at liberty to use the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ (these three vectors are obviously LI, though this is also a theoretical consequence of the fact that the corresponding eigenvalues are *distinct).* Whilst the matrix of f with respect to the canonical basis is A, its matrix with respect to the 'new' basis is

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \tag{1}$$

reflecting the equations

$$A\mathbf{v}_1 = 1\mathbf{v}_1, \qquad A\mathbf{v}_2 = 2\mathbf{v}_1, \qquad A\mathbf{v}_3 = 4\mathbf{v}_3. \tag{2}$$

We shall now make the relationship between the two matrices A and D more explicit.

Let P denote the matrix whose columns are the chosen eigenvectors:

$$P = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix} = \left(\begin{array}{c|c|c} -6 & 1 & 3 \\ 3 & 0 & 0 \\ -5 & 1 & 1 \end{array}\right)$$

(vertical lines emphasize the column structure of this matrix). In view of (2),

$$AP = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ A\mathbf{v}_1 & A\mathbf{v}_2 & A\mathbf{v}_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix} = \left(\begin{array}{c|c|c} -6 & 2 & 12 \\ 3 & 0 & 0 \\ -5 & 2 & 4 \end{array}\right).$$

We get exactly the same result by multiplying P by the diagonal matrix D on the *right:*

$$PD = \left(\begin{array}{c|c|c} -6 & 1 & 3 \\ 3 & 0 & 0 \\ -5 & 1 & 1 \end{array}\right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \left(\begin{array}{c|c|c} -6 & 2 & 12 \\ 3 & 0 & 0 \\ -5 & 2 & 4 \end{array}\right).$$

In conclusion,

$$AP = PD.$$

Since the columns of P are LI, P has rank 3 and is invertible. One can therefore assert (without the need to actually compute P^{-1}) that

$$P^{-1}AP = D, \qquad \text{or} \qquad A = PDP^{-1}.$$

To further make sense of these equations, we record the

Definition. *(i) Two matrices $A, B \in \mathbb{R}^{n,n}$ are said to be* similar *if there exists an invertible matrix P such that $A = PBP^{-1}$.*
(ii) A matrix $A \in \mathbb{R}^{n,n}$ is said to be diagonalizable *if it is similar to a diagonal matrix.*

The property of 'being similar to' is an *equivalence relation* on the set $\mathbb{R}^{n,n}$ (refer to L4.3). The 3×3 matrix A of our example is similar to (1), and therefore diagonalizable.

Warning. We have chosen to apply these definitions strictly within the field $\mathbb{R}$ of real numbers. One is free to treat A, B as elements of $\mathbb{C}^{n,n}$ and ask whether $A = PBP^{-1}$ with $P \in \mathbb{C}^{n,n}$. This is an easier condition to satisfy, and leads to a more general concept of similarity and diagonalizability, but one that we shall ignore in this course.

L21.2 A criterion for diagonalizability. We explain the construction in L20.1 in a more general context. Suppose that $A \in \mathbb{R}^{n,n}$ possesses two eigenvectors $\mathbf{v}_1, \mathbf{v}_2$, with corresponding eigenvalues λ_1, λ_2 that may or may not be distinct. Consider the matrix

$$X = \begin{pmatrix} \uparrow & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 \\ \downarrow & \downarrow \end{pmatrix} \in \mathbb{R}^{n,2}$$

whose two columns are the chosen eigenvectors. Arguing as before,

$$AX = \begin{pmatrix} \uparrow & \uparrow \\ A\mathbf{v}_1 & A\mathbf{v}_2 \\ \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} \uparrow & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 \\ \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = XD,$$

where this time D is a diagonal 2×2 matrix.

The last equation is valid even if the two eigenvectors are identical, and we can choose more eigenvectors so as to obtain a matrix X with more columns. But we can only *invert* X if its rank is n, or equivalently if we can find a total of n LI eigenvectors. Hence the

Lemma. *A matrix $A \in \mathbb{R}^{n,n}$ is diagonalizable iff there exists a* basis *of $\mathbb{R}^n$ consisting of eigenvectors of A.*

The next result tells us exactly when this is possible.

Theorem. *A matrix $A \in \mathbb{R}^{n,n}$ is diagonalizable iff all the roots of $p(x)$ are real, and for each repeated root λ we have*

$$\text{mult}(\lambda) = \dim E_\lambda. \tag{3}$$

Proof. Each eigenvector $\mathbf{v} \in \mathbb{R}^{n,1}$ is associated to a real root λ of $p(x)$, and we already know that the dimension of the eigenspace E_λ is *at most* $\text{mult}(\lambda)$. So unless (3) holds for every eigenvalue, it is numerically impossible to find a basis of eigenvectors.

Conversely, suppose that the distinct roots of $p(x)$ are $\lambda_1, \dots, \lambda_k \in \mathbb{R}$. If $k = n$ the result follows from the Corollary in L19.1; otherwise set $m_i = \text{mult}(\lambda_i)$ and suppose that (3) holds. Pick a basis of each eigenspace E_{λ_i}, and put all these elements together to get a total of $n = m_1 + \cdots + m_k$ eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$. Any linear relation between them can be regrouped into a linear relation

$$a_1\mathbf{w}_1 + \cdots + a_k\mathbf{w}_k = 0, \quad \text{with} \quad a_i \in \mathbb{R} \quad \text{and} \quad A\mathbf{w}_i = \lambda_i\mathbf{w}_i,$$

in which each $\mathbf{w}_i$ is itself a LC of $\mathbf{v}$'s if $m_i > 1$. Since the $\mathbf{w}_i$'s correspond to *distinct* eigenvalues all the coefficients must vanish, and it follows that the $\mathbf{v}$'s form a basis. QED

Exercise. Verify that the 'anti-diagonal' matrix

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

has characteristic polynomial $-(x-1)^2(x+1)$, and eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Deduce that $\text{mult}(1) = 2 = \dim E_1$, and that $P^{-1}BP = D$, where

$$P = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

L21.3 An application. The sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$ of Fibonacci numbers is defined recursively by the initial values $f_0 = 0$, $f_1 = 1$ and the equation

$$f_{n+1} = f_n + f_{n-1}.$$

The latter can be put into matrix form

$$\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = F\begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix} \quad \text{by taking } F = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

The characteristic polynomial of F is $x^2 - x - 1 = (x - \lambda_1)(x - \lambda_2)$, where

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \qquad \lambda_2 = \frac{1-\sqrt{5}}{2} \tag{4}$$

($\lambda_1 = 1.618..$ is the so-called *golden ratio).* It follows that $F = PDP^{-1}$, where

$$P = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}, \qquad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad P^{-1} = \frac{1}{\lambda_2 - \lambda_1}\begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix}.$$

Powers of F are now easily computed:

$$F^n = (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1}) = PD^nP^{-1},$$

since all internal pairs $P^{-1}P$ cancel out. It follows that

$$\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = F^n\begin{pmatrix} 0 \\ 1 \end{pmatrix} = PD^nP^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \tfrac{1}{\sqrt{5}}P\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and we obtain the celebrated formula

$$f_n = \frac{\lambda_1^n - \lambda_2^n}{\sqrt{5}}$$

for the nth Fibonacci number in terms of (4). For large n, this is very close to $\lambda_1^n/\sqrt{5}$ (for instance, $\lambda_1^{12}/\sqrt{5} = 144.001..$). Moreover, the ratio f_{n+1}/f_n tends to λ_1 as $n \to \infty$.

L21.4 Further exercises.

1. Which of the following matrices A is diagonalizable? Find an invertible matrix $P \in \mathbb{R}^{3,3}$ (if it exists) such that $P^{-1}AP$ is diagonal.

$$\begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 4 & -1 \end{pmatrix}.$$

2. Given the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 3 & 2 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

find invertible matrices P_1 and P_2 such that $P_1^{-1}AP_1 = B_1$ and $P_2^{-1}AP_2 = B_2$.

3. Find counterexamples to show that *both* the following assertions are false:

$A \in \mathbb{R}^{n,n}$ is diagonalizable $\Rightarrow$ A is invertible;

$A \in \mathbb{R}^{n,n}$ is invertible $\Rightarrow$ A is diagonalizable.

4. Let $A = \begin{pmatrix} -5 & 3 \\ 6 & -2 \end{pmatrix}$. Find a diagonal matrix D and a matrix P such that $A = PDP^{-1}$. If $D = E^3$, check that $A = (PEP^{-1})^3$; hence find a matrix $B \in \mathbb{R}^{2,2}$ such that $A = B^3$.

5. Let $g\colon \mathbb{R}^{3,3} \to \mathbb{R}^{3,3}$ denote the linear mapping defined by $g(A) = A + A^\top$. Use the study of g carried out in a previous lecture to find a basis of $\mathbb{R}^{3,3}$ consisting of eigenvectors of g. Write down the daigonal 9×9 matrix representing g with respect to this basis.

6. Let $c \in \mathbb{R}$. Prove that $A \in \mathbb{R}^{n,n}$ is diagonalizable if and only if $A + cI$ is.