Notes 20 – Eigenspaces and Multiplicities

In this lecture, we shall explain how to compute methodically *all* the eigenvectors associated to a given square matrix.

L20.1 Subspaces generated by eigenvectors. Given $A \in \mathbb{R}^{n,n}$, consider its characteristic polynomial

$$p(x) = \det(A - xI).$$

We know that this polynomial has degree n, and its roots are precisely the eigenvalues of A: a real number λ satisfies $p(\lambda) = 0$ iff there exists a *nonnull* column vector $\mathbf{v} \in \mathbb{R}^{n,1}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

Definition. *If λ is an eigenvalue, the subspace*

$$E_\lambda = \ker(A - \lambda I) = \{\mathbf{v} \in \mathbb{R}^{n,1} : A\mathbf{v} = \lambda\mathbf{v}\}$$

of $\mathbb{R}^n$ is called the eigenspace *associated to λ.*

𝔚𝔞𝔯𝔫𝔦𝔫𝔤: Not quite all the elements of E_λ are eigenvectors, since (being a subspace) E_λ also includes the null vector $\mathbf{0}$ that is not counted as an eigenvector.

The dimension of E_λ satisfies

$$n - \dim E_\lambda = r(A - \lambda I),$$

by the Rank-Nullity Theorem or (RC2).

Example. The matrix $A = \begin{pmatrix} -6 & 9 \\ -4 & 7 \end{pmatrix}$ has characteristic polynomial

$$p(x) = (-6 - x)(7 - x) + 36 = x^2 - x - 6 = (x + 2)(x - 3).$$

The roots are $\lambda_1 = -2$ and $\lambda_2 = 3$, and we consider one at a time. Firstly,

$$A - \lambda_1 I = A + 2I = \begin{pmatrix} -4 & 9 \\ -4 & 9 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & -\frac{9}{4} \\ 0 & 0 \end{pmatrix}.$$

As predicted (by the very fact that -2 is an eigenvalue), this matrix has rank less than 2. It is easy to find a nonnull vector in $\mathrm{Ker}(A + 2I)$, namely

$$\begin{pmatrix} \frac{9}{4} \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 9 \\ 4 \end{pmatrix}, \qquad \text{whence} \quad E_{-2} = \mathscr{L}\left\{\begin{pmatrix} 9 \\ 4 \end{pmatrix}\right\}.$$

Similarly,

$$A - \lambda_2 I = A - 3I = \begin{pmatrix} -9 & 9 \\ -4 & 4 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & -1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad E_3 = \mathscr{L}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}.$$

Although the previous example only has $n = 2$, it illustrates an important technique: that of selecting an eigenvector for each eigenvalue.

The next result is fairly obvious for $k = 2$, and we prove it in another special case.

Proposition. *Suppose that* $\mathbf{v}_1, \ldots, \mathbf{v}_k$ *are eigenvectors of* A *associated to* distinct *eigenvalues* $\lambda_1, \ldots, \lambda_k$. *Then* $\mathbf{v}_1, \ldots, \mathbf{v}_k$ *are LI.*

Proof. To give the general idea, we take $k = 3$ (which, for the conclusion to be valid, means that the vectors lie in $\mathbb{R}^n$ with $n \geqslant 3$) and $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$. Suppose that

$$0 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3. \tag{1}$$

Applying A (or rather its associated linear mapping) gives

$$0 = a_1A\mathbf{v}_1 + a_2A\mathbf{v}_2 + a_3A\mathbf{v}_3 = a_1\mathbf{v}_1 + 2a_2\mathbf{v}_2 + 3a_3\mathbf{v}_3,$$

and subtracting (1),

$$0 = a_2\mathbf{v}_2 + 2a_3\mathbf{v}_3. \tag{2}$$

Applying A again,

$$0 = a_2A\mathbf{v}_2 + 2a_3A\mathbf{v}_3 = 2a_2\mathbf{v}_2 + 6a_3\mathbf{v}_3.$$

Substracting twice (2) gives $a_3 = 0$, since the eigenvector $\mathbf{v}_3$ is necessarily nonnull. Returning to (2) we get $a_2 = 0$, and finally $a_1 = 0$ from (1). Thus, there is no non-trivial linear relation between the three eigenvectors, and they are LI. QED

Recall that a total of n LI vectors in the vector space $\mathbb{R}^n$ automatically forms a basis.

Corollary. *Suppose that the characteristic polynomial of a matrix* $A \in \mathbb{R}^{n,n}$ *has* n distinct real *roots. Then* $\mathbb{R}^n$ *has a* basis *consisting of eigenvectors of* A.

We shall see in the next lecture that the Corollary's conclusion means that A is what is called *diagonalizable:* in many ways A behaves as if it were a diagonal matrix.

L20.2 Repeated roots. Greater difficulties can arise when a root of $p(x)$ is not simple.

Definition. *We define the* multiplicity, *written* $\mathrm{mult}(\lambda)$, *of a root* λ *of the characteristic polynomial* $p(x)$ *of* $A \in \mathbb{R}^{n,n}$ *to be the power of the factor* $x - \lambda$ *that occurs in* $p(x)$.

Example. We spotted some eigenvectors of the matrix

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

in the previous lecture. Its characteristic polynomial is

$$\begin{aligned} p(x) &= (-2-x)\big[(-2-x)(-2-x)-1\big] - \big[-3-x\big] + \big[3+x\big] \\ &= -x^3 - 6x^2 - 9x = -x(x+3)^2, \end{aligned}$$

whence $\mathrm{mult}(0) = 1$ and $\mathrm{mult}(-3) = 2$. To find the space E_{-3} generated by the eigenvectors with eigenvalue -3, consider

$$A + 3I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This has rank 1, and there is a 2-dimensional space

$$E_{-3} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y + z = 0 \right\} = \mathscr{L}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

of solutions. The fact that $\dim E_{-3} = \mathrm{mult}(-3)$ is not entirely a coincidence.

Theorem. *Let λ be an eigenvalue of a square matrix A. Then*

$$1 \leqslant \dim E_\lambda \leqslant \text{mult}(\lambda).$$

Proof. The first inequality is obvious: the fact that λ *is* an eigenvalue means that E_λ contains a *nonzero* vector $\mathbf{v}$.

To prove the second inequality one needs to know more about the characteristic polynomial, but we can justify it in the special case that the remaining roots of $p(x)$ are real and distinct. Let $A \in \mathbb{R}^{n,n}$, $m = \text{mult}(\lambda)$. Suppose (for a contradiction) that $\dim E_\lambda > m$, so that we can pick LI vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{m+1}$ in E_λ, as well as LI eigenvectors $\mathbf{w}_1, \ldots, \mathbf{w}_{n-m}$, one for each remaining eigenvalue. The resulting total of $n+1$ vectors in $\mathbb{R}^n$ cannot be LI, so there is a *non-trivial* linear relation

$$\underbrace{a_1\mathbf{v}_1 + \cdots + a_{m+1}\mathbf{v}_{m+1}}_{\mathbf{v}} + b_1\mathbf{w}_1 + \cdots + b_{n-m}\mathbf{w}_{n-m} = \mathbf{0}.$$

But one way or another this contradicts the Proposition: the sum $\mathbf{v}$ is nonnull since the $\mathbf{w}$'s are LI, and is itself an eigenvector with eigenvalue λ different from the others. QED

Exercise. (i) Write down a diagonal matrix $A \in \mathbb{R}^{4,4}$ such that the charactristic polynomial of A equals $(x-1)^2(x+2)^2$.

(ii) Verify that A has $\dim E_1 = 2$ and $\dim E_{-2} = 2$.

(iii) Let B be the matrix obtained from A by changing its entry in row 1, column 2 from 0 to 1. Compute $\dim E_1$ and $\dim E_{-2}$ for B.

(iv) Find a matrix C with the same characteristic polynomial $(x-1)^2(x+2)^2$ but for which $\dim E_1 = 1 = \dim E_{-2}$.

In the light of the Theorem, some authors call
$\text{mult}(\lambda)$ the *algebraic multiplicity* of λ, and $\dim(E_\lambda)$ the *geometric multiplicity* of λ.
The latter can never exceed the former.

L20.3 The 2×2 case. Let us analyse the possible eigenspaces of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2,2}.$$

Its characteristic polynomial $p(x) = x^2 - (a+d)x + ad - bc$ has roots

$$\frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = \frac{a+d \pm \sqrt{\Delta}}{2},$$

where

$$\Delta = (a-d)^2 + 4bc.$$

Consider the three cases:
(i) If $\Delta > 0$ then there are distinct real eigenvalues λ_1, λ_2 so that

$$p(x) = (x-\lambda_1)(x-\lambda_2), \qquad \lambda_1+\lambda_2 = a+d, \quad \lambda_1\lambda_2 = ad+bc.$$

Any associated pair of eigen*vectors* will form a basis of $\mathbb{R}^2$, by the Corollary.

(ii) If $\Delta = 0$ there is one eigenvalue λ with $\mathrm{mult}(\lambda) = 2$. In the subcase that $\dim E_\lambda = 2$, $E_\lambda = \mathbb{R}^2$ contains both $\mathbf{e}_1, \mathbf{e}_2$ and

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I, \tag{3}$$

which means that $b = c = 0$ and $a = d$.

(iii) If $\Delta < 0$ there are no real eigenvalues, though the theory still makes sense when one passes to the field $F = \mathbb{C}$ of complex numbers (see q. 5 below).

An important special case (of (i) or (ii)) is that in which $b = c$, and A is *symmetric:* this implies that $\Delta \geqslant 0$ with equality iff A is given by (3). Actually, one can prove that *the eigenvalues of any real symmetric $n \times n$ matrix are always themselves real.*

Example. An instance of (ii) with $\dim E_\lambda = 1$ is the matrix $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ that satisfies $N^2 = 0$. The characteristic polynomial is x^2, so 0 is a repeated eigenvalue. A direct way of seeing that any eigenvalue must be 0 is to observe that

$$N\mathbf{v} = \lambda\mathbf{v} \quad \Rightarrow \quad \mathbf{0} = N^2\mathbf{v} = N(\lambda\mathbf{v}) = \lambda^2\mathbf{v} \quad \Rightarrow \quad \lambda^2 = 0.$$

Any eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$ has to satisfy $y = 0$, so $E_0 = \mathscr{L}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ has dimension 1. Similar considerations apply to the matrix $N + aI$ where $a \neq 0$.

L20.4 Further exercises.

1. For each of the following matrices, find all possible eigenvalues $\lambda \in \mathbb{R}$ and describe the associated eigenspaces E_λ:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 5 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 & 2 \\ -1 & 3 & 0 & 0 \\ -1 & 4 & -1 & 1 \\ -1 & 4 & -1 & 0 \end{pmatrix}.$$

2. Given $A = \begin{pmatrix} a & 2 & a-1 \\ -3 & 5 & -2 \\ -4 & 4 & -1 \end{pmatrix}$ with $a \in \mathbb{R}$,

(i) find the value of a for which 1 is an eigenvalue (no need to work out $p(x)$!).

(ii) is there a value of a for which there are two LI eigenvectors sharing the same eigenvalue?

3. Let $p(x)$ be a polynomial of degree n, and assume that $p(\lambda) = 0$. Show that $\mathrm{mult}(\lambda) \geqslant 2$ if and only if $p'(\lambda) = 0$.

4. Find a matrix L such that $L^2 = L$ but L is neither 0 nor I. What are the eigenvalues of L? Does the answer depend upon you choice?

5. [Uses the complex field $\mathbb{C}$.] Consider the matrix

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(i) Verify that M satisfies $M^2 + I = 0$ and has characteristic polynomial $x^2 + 1$.

(ii) Show that $M - iI$ and $M + iI$ both have rank 1 (in the obvious sense in which this applies to complex matrices), and find nonnull $\mathbf{u}, \mathbf{v} \in \mathbb{C}^2$ such that $M\mathbf{u} = i\mathbf{u}$ and $M\mathbf{v} = -i\mathbf{v}$.