# Notes 2 – Square matrices and determinants

The study of square matrices is particularly rich, since ones of the same size can be multiplied together repeatedly. This realization will lead us to construct inverse matrices and define a number called the 'determinant' of a square matrix.

---

**L2.1 Identity matrices.** Recall that a matrix is said to be *square* if it has the same number of rows and columns. So $A \in \mathbb{R}^{m,n}$ is square iff $m = n$.

**Definition.** *A square matrix is* diagonal *if the only entries $a_{ij}$ that are nonzero are those for which $i = j$. These form the diagonal $\searrow$ from top left to bottom right. The $n \times n$ matrix $A$ for which*

$$a_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

*is called the* identity matrix *of order $n$, and is denoted $I_n$.*

It is easy to verify the

**Proposition.** *If $A \in \mathbb{R}^{m,n}$ then $I_m A = A = A I_n$.*

Here is an example:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If $A, B \in \mathbb{R}^{n,n}$ then both $AB$ and $BA$ are defined and have size $n \times n$. In general they are unequal, so matrix multiplication is not *commutative.*

> **Exercise.** Try $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = A^{\top}$.

---

**L2.2 Powers of matrices.** We can raise a square matrix to any positive power. For example $A^2$ simply means $AA$, and

$$A^3 = AAA = A^2 A = AA^2.$$

An important property of powers of a given matrix is that they *commute* with one another, i.e. the order of multiplication does not matter (unlike for general pairs of matrices):

$$A^m A^n = A^m A^n, \qquad m, n \in \mathbb{N}. \tag{1}$$

By convention, for a matrix $A \in \mathbb{R}^{n,n}$, we set $A^0 = I_n$. We can try to define negative powers using the *inverse* of a matrix, though this does not always exist. The situation for $n = 2$ is described by the

**Lemma.** *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then there exists $B \in \mathbb{R}^{2,2}$ for which $AB = I_2$ iff $ad - bc \neq 0$. In this case, the same matrix $B$ satisfies $BA = I_2$.*

*Proof.* The recipe is well known:

$$B = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{2}$$

Provided $ad - bc = 0$ this matrix satisfies both the required equations. If $ad - bc = 0$ then

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} A = \mathbf{0} \tag{3}$$

is the null matrix, and this precludes the existence of a $B$ for which $AB = I_2$; multiplying (3) on the right by $B$ would give $a, b, c, d$ are zero, impossible. QED

If $ad - bc \neq 0$, then (2) is called the *inverse* of $A$ and denoted $A^{-1}$. More generally, a square matrix $A \in \mathbb{R}^{n,n}$ is said to be *invertible* or *nonsingular* if there exists a matrix $A^{-1}$ such that $AA^{-1} = I_n$ or $AA^{-1} = I_n$. In this case, it is a remarkable fact that there is only one inverse matrix $A^{-1}$ and it satisfies both equations.

**Exercise.** (i) If $A$ is invertible, then so is $A^\top$, and $(A^\top)^{-1} = (A^{-1})^\top$.

(ii) If $A, B$ are invertible then $(AB)^{-1} = B^{-1}A^{-1}$.

(iii) If $A$ is invertible and $n \in \mathbb{N}$ then $(A^n)^{-1} = (A^{-1})^n$.

The inverse can be used to help solve equations involving square matrices. For example, suppose that

$$AB = C,$$

where $A$ is an invertible square matrix. Then

$$B = I_n B = (A^{-1}A)B = A^{-1}(AB) = A^{-1}C,$$

and we have solved for $B$ in terms of $C$.

**Example.** Let $A \in \mathbb{R}^{2,2}$. A direct calculation shows that

$$A^2 - (a+d)A + (ad - bc)I_2 = 0.$$

Assuming there exists $A^{-1}$ such that $AA^{-1} = I_2$ we obtain

$$A - (a+d)I_2 + (ad - bc)A^{-1} = 0 \quad \Rightarrow \quad (ad - bc)A^{-1} = (a+d)I_2 - A.$$

We get exactly the same expression for $A^{-1}$ by assuming that $A^{-1}A = I_2$.

---

**L2.3 Determinants.** The quantity $ad - bc$ is called the *determinant* of the $2\times 2$ matrix $A$. It turns out that it is possible to associate to any square matrix $A \in \mathbb{R}^{n,n}$ a number called its *determinant*, written $\det A$ or $|A|$. This number is a function of the components of $A$, and satisfies

**Theorem.** $\det A \neq 0$ *iff* $A$ *is invertible.*

We shall explain this result in Part II of the course, but here we give two ways of computing the determinant when $n = 3$. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Then one copies down the first two columns to form the extended array

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}.$$

The formula of *Sarrus* asserts that the determinant of $A$ is the sum of the products of entries on the three downward diagonals $\searrow$ minus those on the three upward diagonals $\nearrow$.

Equivalently,

$$\det A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \tag{4}$$

The three mini-determinants are constructed from the last two rows of $A$.

**Exercise.** Use either of these formulae to prove the following properties for the determinant of a $3{\times}3$ matrix $A$:

(i) if one row is multiplied by $c$ so is $\det A$,

(ii) $\det(cA) = c^3 \det A$,

(iii) if two rows are swapped then $\det A$ changes sign,

(iv) if one rows is a multiple of another then $\det A = 0$,

(v) $\det A = \det(A^\top)$, so the above statements apply equally to columns.

In order to explain where (4) comes from, let $A_{ij} \in \mathbb{R}^{2,2}$ denote the matrix obtained from $A$ by deleting its $i$ row and $j$ column. Let $\widetilde{A}$ denote the matrix with entries

$$\widetilde{a}_{ij} = (-1)^{i+j}\det(A_{ij}).$$

In words, it is formed by replacing each entry of $A$ by the determinant of its 'complementary matrix' changing sign chess-board style.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

**Proposition.** *$A\widetilde{A}^\top = (\det A)I_n$, so if $\det A \neq 0$ is nonzero then $A^{-1} = \dfrac{1}{\det A}\widetilde{A}^\top$.*

We shall prove this result after introducing the vector cross product, but note that the right-hand side of (4) is the top-left entry of $A\widetilde{A}^\top$. he matrix $\widetilde{A}^\top$ is called the *adjoint* or *adjugate* of $A$.

**Example.** To find the inverse of

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 5 & 8 \\ 13 & 21 & 35 \end{pmatrix},$$

we compute

$$A\widetilde{A} = A\begin{pmatrix} 7 & -1 & -2 \\ 7 & 9 & -8 \\ -2 & -2 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 5 & 8 \\ 13 & 21 & 35 \end{pmatrix}\begin{pmatrix} 7 & 7 & -2 \\ -1 & 9 & -2 \\ -2 & -8 & 2 \end{pmatrix} = 2I_3.$$

Thus $\det A = 2$ and

$$A^{-1} = \begin{pmatrix} \frac{7}{2} & \frac{7}{2} & -1 \\ -\frac{1}{2} & \frac{9}{2} & -1 \\ -1 & -4 & 1 \end{pmatrix}.$$

One can also check that $A^{-1}A = I_3$.

## L2.4 Further exercises.

1. Compute the following products of matrices:

$$\begin{pmatrix} 6 & 0 & 4 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 4 \\ -2 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 0 & 4 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 6 & 0 & 4 \\ 1 & -1 & 1 \end{pmatrix}.$$

2. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \\ 3 & 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 0 & 4 \\ 1 & 0 & 0 \\ 3 & 2 & 5 \end{pmatrix}$. Compute the following products:

$$AB, \quad BA, \quad (BA)^\top, \quad (AB)^\top, \quad A^\top B^\top, \quad B^\top A^\top.$$

3. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 6 & 9 \end{pmatrix}, \qquad B = \begin{pmatrix} a & b \\ 0 & c \\ 0 & 0 \end{pmatrix}.$$

Find $a, b, c$ so that $AB = I_2$. Does there exist a matrix $C$ such that $CA = I_3$?

4. The matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ satisfy the equations $AE = EA$, $AE^\top = E^\top A$. Deduce that $A = aI_2$.

5. Give examples of matrices $A, B, C \in \mathbb{R}^{2,2}$ for which
   (i) $A^2 = 0$ and $A \neq 0$,
   (ii) $B^2 = B$ and $B \neq I_2$,
   (iii) $C^2 + 2C + I_2 = 0$.

6. Find the inverses of the following matrices:

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ -1 & 4 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 3 & 2 & 1 & 1 \end{pmatrix}.$$

For the last one, you will need to generalize the chess-board method.

7. Let $A = \begin{pmatrix} 5 & 0 & -1 \\ 0 & 5 & 1 \\ -1 & 1 & 4 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$. Compute the following matrices:

$$AP, \quad P^{-1}, \quad D = P^{-1}AP, \quad PD, \quad AD.$$