Notes 18 – Eigenvectors and Eigenvalues

L18.1 Eigenvectors of a linear transformation. Consider a linear mapping

$$f: V \to V,$$

where V is a vector space with field of scalars F.

Definition. *A nonzero element* $\mathbf{v} \in V$ *is called an* eigenvector *of* f *if there exists* λ *(possibly 0) in* F *such that*

$$f(\mathbf{v}) = \lambda\mathbf{v}. \tag{1}$$

In these circumstances, λ *is called the* eigenvalue *associated to* $\mathbf{v}$*.*

$\mathfrak{Warning}$: Since $f(\mathbf{0}) = \mathbf{0}$, it is obvious that the null vector satisfies (1). But the null vector does *NOT* count as an eigenvector; for one thing its eigenvalue λ is undetermined. On the the hand, observe that if $\mathbf{v}$ is an eigenvector of f and $a \neq 0$ then $a\mathbf{v}$ is also an eigenvector (with the same eigenvalue).

Example. Here are two extreme cases:

(i) Suppose f is the *identity* mapping, so that $f(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$. This is obviously linear, and every nonnull vector $\mathbf{v} \in V$ is an eigenvector with eigenvalue 1.

(ii) Define $g(\mathbf{v}) = \mathbf{0}$ for every $\mathbf{v} \in V$ (the 'null' transformation, linear by default). Once again, every nonnull $\mathbf{v} \in V$ is an eigenvector, but this time with common eigenvalue 0.

More interesting examples of eigenvectors can easily be written down if there is a *basis* of the vector space V at one's disposal:

Example. (i) Take $V = \mathbb{R}^2$ and define a linear mapping $f: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(\mathbf{e}_1) = \mathbf{e}_1, \qquad f(\mathbf{e}_2) = -\mathbf{e}_2.$$

By this very definition, $\mathbf{e}_1$ and $\mathbf{e}_2$ are eigenvectors with eigenvalues 1 and -1. Geometrically, f is a *reflection* in the x-axis, and any reflection in the plane will have two such eigenvectors.

(ii) Suppose that V has dimension 4 and a basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. We are at liberty to define

$$f(\mathbf{v}_1) = \mathbf{v}_2, \quad f(\mathbf{v}_2) = \mathbf{v}_3, \quad f(\mathbf{v}_3) = \mathbf{v}_4, \quad f(\mathbf{v}_4) = \mathbf{v}_1,$$

and this uniquely determines the linear mapping $f: V \to V$ for which

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4, \qquad \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 - \mathbf{v}_4$$

are eigenvectors with respective eigenvalues 1 and -1.

To show that bases are not essential for the existence of eigenvectors, here is an example in which the vector space is not finite-dimensional:

Exercise. Let V be the vector space of functions $\phi: \mathbb{R} \to \mathbb{R}$ that admit derivatives of *all* orders. Then the mapping $D: V \to V$ given by

$$D(\phi) = \phi', \quad \text{where} \quad \phi'(x) = \frac{d\phi}{dx},$$

is linear. Find all the eigenvectors (or rather, 'eigenfunctions') of D.

L18.2 Eigenvectors of a square matrix. Suppose that V is a vector space of finite dimension n with $F = \mathbb{R}$ (we shall only consider this case from now on). Once we have chosen a basis of V (any one will have n elements), we know that we can treat V as if it were $\mathbb{R}^n$. For this reason, we need only study linear mappings $f\colon \mathbb{R}^n \to \mathbb{R}^n$, any one of which is given by

$$f(\mathbf{v}) = A\mathbf{v},$$

where $A \in \mathbb{R}^{n,n}$ is a *square* matrix.

Accordingly, an eigenvector of the matrix A is a nonzero *column vector* $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{2}$$

In some lucky cases, one can find such vectors by inspection.

Example. Consider the matrix

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

It is easy to spot the eigenvector

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \text{with eigenvalue } 0.$$

It is a little less obvious, but nonetheless true, that

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad \text{are all eigenvectors with eigenvalue } -3.$$

However, these three are *not* linearly independent (the second is the first plus the third). Overall, in this example, we can find three eigenvectors that are LI. We shall see later that any square matrix of size $n \times n$ has at most n eigenvectors that are LI.

Let $A \in \mathbb{R}^{n,n}$, and let $I = I_n$ be the identity matrix of the same size. Here is a useful trick:

Lemma. *If $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then it is an eigenvector of $A + aI$ with eigenvalue $\lambda + a$.*

Proof. Immediate, since $(A + aI)\mathbf{v} = A\mathbf{v} + a\mathbf{v} = (\lambda + a)\mathbf{v}$. QED

Exercise. (i) Find two linearly independent eigenvectors of

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Hint: refer to an example on the previous page.

(ii) Let V be a vector space with $F = \mathbb{R}$, and basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ so $\dim V = 2$. A linear mapping $f\colon V \to V$ is defined by setting $f(\mathbf{v}_1) = \mathbf{v}_2$ and $f(\mathbf{v}_2) = -\mathbf{v}_1$. Write down the associated matrix M_f, and show that it has no eigenvector in $\mathbb{R}^{2,1}$.

L18.3 The characteristic polynomial. This is a great tool for detecting eigenvalues.

Lemma. *If* $\mathbf{v}$ *is an eigenvector of* A *with eigenvalue* λ *then* $\det(A - \lambda I) = 0$*. Conversely, if* $\det(A - \lambda I) = 0$ *then there* exists *an eigenvector of* A *with eigenvalue* λ*.*

Proof. Equation (2) can be rewritten as

$$(A - \lambda I)\mathbf{v} = 0 \quad \text{or equivalently} \quad \mathbf{v} \in \operatorname{Ker}(A - \lambda I). \tag{3}$$

Given a solution $\mathbf{v} \neq \mathbf{0}$, the nullity of the matrix $A - \lambda I$ will be nonzero. By (RC2) or the Rank-Nullity Theorem, $r(A - \lambda I) < n$. Thus, $A - \lambda I$ is not invertible and its determinant is necessarily 0.

Conversely, $\det(A - \lambda I) = 0$ implies that $r(A - \lambda I) < n$ and $\operatorname{Ker}(A - \lambda I)$ contains a nonnull vector $\mathbf{v}$. The latter satisfies $A\mathbf{v} = \lambda\mathbf{v}$. QED

Definition/Proposition. *Given* $A \in \mathbb{R}^{n,n}$*, the function*

$$p(x) = \det(A - xI)$$

is called the characteristic polynomial *of the square matrix* A*. It is indeed a* polynomial *of degree* n *in the variable* x*.*

To appreciate this, observe that the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has characteristic polynomial

$$p(x) = \det \begin{pmatrix} a-x & b \\ c & d-x \end{pmatrix} = x^2 - (a+d)x + ad - bc. \tag{4}$$

For any square matrix, the constant term of the characteristic polynomial is given by

$$p(0) = \det(A - 0x) = \det A.$$

At the other extreme, the leading term is always $(-x)^n = (-1)^n x^n$. It is also easy to show that the coefficient of x^{n-1} in $p(x)$ equals $(-1)^{n-1} \operatorname{tr} A$, where $\operatorname{tr} A$ denotes the sum of the diagonal entries. Warning: Some authors (including Wikipedia) define the characteristic polynomial to be $\det(xI - A)$; this is always monic (good!) but attempts to calculate it by hand are usually fraught with sign errors (bad¡).

The statement 'λ is an eigenvalue of A' means that there exists $\mathbf{v} \neq \mathbf{0}$ satisfying (2). Any such λ must therefore be a root of $p(x)$, and a solution of the *characteristic equation*

$$p(x) = 0. \tag{5}$$

Here is an important example:

Lemma. A *is itself invertible iff* 0 *is* not *an eigenvalue of* A*.*

Proof. A is invertible iff $\det A \neq 0$ iff $x = 0$ is not a solution of $\det(A - xI) = 0$. QED

> **Exercise.** Give a different proof of the Lemma, avoiding any mention of the determinant and characteristic polynomial.

If $A \in \mathbb{R}^{n,n}$, then (5) can have at most n roots, and there are at most n distinct eigenvalues. There may be less, as the roots of a polynomial can be repeated, and it is also possible that pairs of roots occur as complex numbers.

Given A, suppose that λ is a root of the characteristic polynomial, so that $p(\lambda) = 0$. From above, we know that there must exist a nonnull column vector $\mathbf{v}$ satisfying (3). We can find therefore such a $\mathbf{v}$ by solving the homogeneous linear system associated to $A - \lambda I$. This we shall do in the next lecture (but if you are desperate, do q. 6 below).

Example. We have not spoken much about 4×4 determinants, but these can be expanded along a row or column into a linear combination of 3×3 determinants. Let

$$A = \begin{pmatrix} x & 0 & 0 & s \\ -1 & x & 0 & r \\ 0 & -1 & x & q \\ 0 & 0 & -1 & x+p \end{pmatrix}.$$

Expanding down the first column,

$$\begin{aligned} \det A &= x \det\begin{pmatrix} x & 0 & r \\ -1 & x & q \\ 0 & -1 & x+p \end{pmatrix} + 1. \det\begin{pmatrix} 0 & 0 & s \\ -1 & x & q \\ 0 & -1 & x+p \end{pmatrix} + 0 + 0 \\ &= x\Big(x(x(x+p)+q) + 1.r\Big) + s \\ &= x^4 + px^3 + qx^2 + rx + s. \end{aligned}$$

This type of example can be used to show that *any* polynomial whose leading term is $(-x)^n$ is the characteristic polynomial of some $n \times n$ matrix.

L18.4 Further exercises.

1. Every eigenvector of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is a LC of $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. True or false?

2. Given $A = \begin{pmatrix} 0 & 2 & a \\ 2 & 1 & 1 \\ a & 1 & 1 \end{pmatrix}$ with $a \in \mathbb{R}$, find the values of a for which $r(A) = 2$. Find the eigenvalues of A when $a = 2$.

3. Find all the eigenvalues of the matrices

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 8 \\ 0 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & 3 \\ 3 & 4 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. Try to find two LI vectors $\mathbf{u}, \mathbf{v}$ that are eigenvectors of *both* $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

5. Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, verify that $A^2 - (a+d)A + (ad-bc)I_2$ is the null matrix.

6. Find all the eigenvectors of the first two matrices in q. 3 by solving appropriate linear systems (one for each eigenvalue).