Notes 17 – Operations on Subspaces

Subspaces of vector spaces (including $\mathbb{R}^n$) can now be conveniently defined as the kernels or images of linear mappings between vector spaces. This leads us to discuss their properties in more detail, and compute their dimensions.

L17.1 Intersections and unions. Let W be any finite-dimensional vector space over a field F (or, if the student prefers) merely $\mathbb{R}^n$ with $F = \mathbb{R}$. Let U, V be two subspaces of the fixed vector space W.

Lemma. *(i) The intersection $U \cap V$ is* always *a subspace of W.*

(ii) The union $U \cup V$ is only *a subspace of W if $U \subseteq V$ or $V \subseteq U$ (in which case of course, it equals V or U).*

Proof. (i) If $\mathbf{w}_1, \mathbf{w}_2 \in U \cap V$ and $a \in F$ then $a\mathbf{w}_1 + \mathbf{w}_2$ belongs to both U and V by assumption. This is a combined verification of (LM1) and (LM2).

(ii) If $U \cup V$ is a subspace, but neither U nor V is contained in the other, then we can choose $\mathbf{u} \in U \setminus V$ and $\mathbf{v} \in V \setminus U$. By assumption, $\mathbf{w} = \mathbf{u} + \mathbf{v} \in U \cup V$ so $\mathbf{w}$ belongs to either U or V. In the former case, $\mathbf{v} = \mathbf{w} - \mathbf{u} \in U$, contradiction. Latter case, similarly. QED

Re-iterating (i), the intersection of *any* number of subpsaces is *always* a subspace. All subspaces contain the null vector $\mathbf{0}$, so at 'worst' this subspace will be $\{\mathbf{0}\}$.

As for unions, there will always exist a smallest sub*space* of W containing $U \cup V$. Any such subspace must certainly contain all the vectors

$$\mathbf{u} + \mathbf{v}, \qquad \text{for any } \mathbf{u} \in U,\ \mathbf{v} \in V, \tag{1}$$

by property (S1) of a subspace. But the set of all these simple sums *is* a subspace:

Definition/Lemma. *Let W be a vector space. The* sum *of two subspaces U, V of W is the set, denoted $U + V$, consisting of all the elements in (1). It is a subspace, and is contained inside any subspace that contains $U \cup V$.*

Proof. Typical elements of $U + V$ are $\mathbf{u}_1 + \mathbf{v}_1$ and $\mathbf{u}_2 + \mathbf{v}_2$ with $\mathbf{u}_i \in U$ and $\mathbf{v}_i \in V$. Their sum is

$$(\mathbf{u}_1 + \mathbf{v}_1) + (\mathbf{u}_2 + \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2) \ \in\ U + V.$$

Similarly,

$$a(\mathbf{u}_1 + \mathbf{v}_1) = (a\mathbf{u}_1) + (a\mathbf{v}_1) \ \in\ U + V.$$

This is all we need to affirm that $U + V$ is a subspace.

If X is a subspace that contains $U \cup V$ then it has to contain all elements $u \in U$ and $v \in V$, and *therefore* all elements $u + v \in U + V$. It therefore contains $U + V$. QED

One also says that $U + V$ is the *subspace generated by U and V*. This actually gives a clearer idea of its definition.

In practice, $U + V$ contains any LC of elements drawn from U and V, because such a LC can always be re-arranged into the form (1). We can also think of $U + V$ as the intersection of *all* (typically infinitely many) subspaces containing both U and V.

Example. Consider the subspaces $U = \mathscr{L}\{\mathbf{e}_1, \mathbf{e}_2\}$ and $V = \mathscr{L}\{\mathbf{e}_3, \mathbf{e}_4\}$ of $\mathbb{R}^4$. Then $U + V = \mathbb{R}^4$ because *any* vector in $\mathbb{R}^4$ can be expressed as the sum

$$\underbrace{a_1\mathbf{e}_1 + a_2\mathbf{e}_2}_{\mathbf{u}\in U} + \underbrace{a_3\mathbf{e}_3 + a_4\mathbf{e}_4}_{\mathbf{v}\in V} \quad = \quad \mathbf{u} + \mathbf{v}.$$

This situation is somewhat special, as it is also the case that $U \cap V = \{\mathbf{0}\}$. Lots of similar examples (of sums of two subspaces with zero intersection) can be constructed in any vector space, once one has a basis to play with.

L17.2 Visualizing subspaces in $\mathbb{R}^2$ and $\mathbb{R}^3$. We can represent the vector space $\mathbb{R}^2$ by points of the plane, in which the null vector $\mathbf{0}$ corresponds to the origin. An arbitrary vector in $\mathbb{R}^2$ is represented by the tip of the arrow it defines, placed at the origin. In this way, it is obvious that any subspace U of $\mathbb{R}^2$ is either

(i) the origin itself, corresponding to the zero subspace $\{\mathbf{0}\}$,
(ii) any straight line passing through the origin,
(iii) the whole plane, corresponding to $\mathbb{R}^2$.

In these cases, the dimension of U is respectively $0, 1, 2$.

If U_1, U_2 are two distinct subspaces of $\mathbb{R}^2$, each of dimension 1, they are both represented by lines containing the origin. One easily sees that

$$U_1 \cap U_2 = \{\mathbf{0}\} \qquad \text{and} \qquad U_1 + U_2 = \mathbb{R}^2.$$

The last equality follows because *any* vector in $\mathbb{R}^2$ can be expressed as the *sum* of something in U_1 with something in U_2. (The most obvious case is that in which $U_1 = \mathscr{L}\{\mathbf{e}_1\}$ and $U_2 = \mathscr{L}\{\mathbf{e}_2\}$ correspond to the two axes and $(x, y) = x\mathbf{e}_1 + y\mathbf{e}_2 \in U_1 + U_2$.)

We can carry out a similar analysis for subspaces V in $\mathbb{R}^3$, representing the latter by points in space. In this situation, $\dim V = 1$ again gives rise to a straight line through the origin, but $\dim V = 2$ gives any plane passing through the origin. If V_1, V_2 are two such distinct 2-dimensional subspaces (planes through the origin), one easily sees this time that

$$V_1 \cap V_2 = V_3 \qquad \text{and} \qquad V_1 + V_2 = \mathbb{R}^3,$$

where V_3 is a line (again containing the origin). Note that in this last case,

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2).$$

We shall see that this formula holds in general.

L17.3 Dimension counting. Any subspace U is a vector space in its own right and has a *dimension:* recall that this equals the number of elements inside *any* basis of U.

Obvious lemma. *If U is a subspace of a vector space (or another subspace) V then* $\dim U \leqslant \dim V$*, with equality iff $U = V$.*

This is true because a basis of U can always be extended until it becomes one of V. To do this we can use the trick that if $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are LI and $\mathbf{v}_{k+1}$ is *not* a LC of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, then $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}$ are LI. In the examples above, a subspace of $\mathbb{R}^2$ has dimension 2 only if it *is* $\mathbb{R}^2$. Similarly for dimension 3 in $\mathbb{R}^3$.

Much of the theory of bases and dimension was discovered by Hermann Grassmann, including the following result dating from around 1860:

Theorem. *Let U, V be two subspaces of a finite-dimensional vector space W. Then*

$$\dim U + \dim V = \dim(U \cap V) + \dim(U + V). \tag{2}$$

This result is illustrated by the following example (whose method is often used as a proof).

Example. Let $W = \mathbb{R}^5$. Consider the two subspaces

$$\begin{aligned} U &= \mathscr{L}\{(x_1, x_2, x_3, x_4, x_5) : 2x_1 - x_2 - x_3 = 0 = x_4 - 3x_5\}, \\ V &= \mathscr{L}\{(x_1, x_2, x_3, x_4, x_5) : x_3 + x_4 = 0\}. \end{aligned}$$

We are required to find a basis of $\mathbb{R}^5$ that contains *both* a basis of U *and* a basis of V. The trick is to start by finding a basis of $U \cap V$. It is easy to see that $\dim U = 3$ and $\dim V = 4$; this is because the homogeneous linear systems have rank 2 and 1. Now, a vector $\mathbf{v} \in \mathbb{R}^5$ belongs to $U \cap V$ iff it satisfes *all three* equations. Since the associated matrix

$$A = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & -\frac{1}{2} & 0 & 0 & \frac{3}{2} \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & -3 \end{pmatrix}$$

has rank 3, we deduce that $\dim(U \cap V) = 5 - 3 = 2$. Indeed, we may take $x_2 = s$ and $x_5 = t$ to be free variables and obtain (as a row) the general solution

$$\mathbf{v} = \left(\tfrac{1}{2}s - \tfrac{3}{2}t,\ s,\ -3t,\ 3t,\ t\right).$$

A basis of $U \cap V$ consists of

$$\mathbf{w}_1 = (\tfrac{1}{2}, 1, 0, 0, 0), \qquad \mathbf{w}_2 = (-\tfrac{3}{2}, 0, -3, 3, 1)$$

(take first $s = 1$, $t = 0$ and second $s = 0$, $t = 1$). Extend this basis in any way to

a basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of U, and

a basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4, \mathbf{w}_5\}$ of V, and

Then $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\}$ will *always* be LI and thus a basis of $\mathbb{R}^5$. There are lots of choices in this example, but we could take

$\mathbf{w}_3 = (0, -1, 1, 0, 0)$ (this works since $\mathbf{w}_3 \in U$ but $\mathbf{w}_3 \notin \mathscr{L}\{\mathbf{w}_1, \mathbf{w}_2\}$),

$\mathbf{w}_4 = (0, 0, 1, -1, 0)$, $\mathbf{w}_5 = (0, 0, 0, 0, 1)$ (note that $\mathbf{w}_5 \notin \mathscr{L}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\}$).

In conclusion, $U + V = \mathbb{R}^5$, and the required basis is

$$\overbrace{\mathbf{w}_5 \quad \rlap{$\underbrace{\phantom{\mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3 \quad \mathbf{w}_4}}_{V}$}\mathbf{w}_1 \quad \mathbf{w}_2}^{U} \quad \mathbf{w}_3 \quad \mathbf{w}_4$$

Fancy proof of (2). First consider the *Cartesian product* $P = U \times V$ consisting of ordered pairs $(\mathbf{u}, \mathbf{v})$ with $\mathbf{u} \in U$ and $\mathbf{v} \in V$. This can be made into a vector space using the operations

$$\begin{aligned} (\mathbf{u}_1, \mathbf{v}_1) + (\mathbf{u}_2, \mathbf{v}_2) &= (\mathbf{u}_1 + \mathbf{u}_2,\ \mathbf{v}_1 + \mathbf{v}_2), \\ a(\mathbf{u}, \mathbf{v}) &= (a\mathbf{u},\ a\mathbf{v}). \end{aligned}$$

If $\mathbf{u}_1, \ldots, \mathbf{u}_m$ is a basis of U and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ a basis of V, it is easy to verify that

$$(\mathbf{u}_1, \mathbf{0}),\ \ldots\ , (\mathbf{u}_m, \mathbf{0}),\ (\mathbf{0}, \mathbf{v}_1),\ \ldots\ , (\mathbf{0}, \mathbf{v}_n)$$

is a basis of P. Hence, $\dim P = m+n$ (P is called the *external direct sum* of U and V). Consider the mapping

$$f\colon P \to W, \qquad f(\mathbf{u},\mathbf{v}) = \mathbf{u}+\mathbf{v}.$$

One easily checks that (i) f is linear, (ii) the image of f equals $U+V$, and (iii) the kernel of f equals $\{(\mathbf{w},-\mathbf{w}) : \mathbf{w}\in U\cap V\}$. Since the last subspace has the same dimension as $U\cap V$, the Theorem follows from the Rank-Nullity formula: $\dim P = \dim\operatorname{Ker} f + \dim\operatorname{Im} f$. QED

Example. Consider two subspaces $U = \mathscr{L}\{\mathbf{u}_1,\mathbf{u}_2\}$, $V = \mathscr{L}\{\mathbf{v}_1,\mathbf{v}_2\}$ of $\mathbb{R}^n$. There are two competing ways to decide mechanically whether $U=V$:

(i) Super-reduce the $2\times n$ matrix with rows $\mathbf{u}_1,\mathbf{u}_2$. Do the same for $\mathbf{v}_1,\mathbf{v}_2$. Then $U=V$ iff the two super-reduced matrices are identical. (This method works because the super-reduced version of a matrix is unique.)

(ii) Step-reduce the $4\times n$ matrix with rows $\mathbf{u}_1,\mathbf{u}_2,\mathbf{v}_1,\mathbf{v}_2$ to find its rank ρ. Then $U=V$ iff $\rho=2$. (This works because in general $\rho=\dim(U+V)$, whereas $U=V$ iff $U+V=U=V$.)

L17.4 Further exercises.

1. Given the subspaces

$$U = \{(0,a,b,c) : a,b,c\in\mathbb{R}\}, \qquad V = \{(p,q,p,r) : p,q,r\in\mathbb{R}\}$$

of $\mathbb{R}^4$, find a basis of each of U, V, $U\cap V$, $U+V$.

2. In $\mathbb{R}^5$, find a basis for the intersection of the two subspaces

$$\begin{aligned} U_1 &= \{(x_1,x_2,x_3,x_4,x_5) : 2x_1-x_2-x_3 = 0 = x_4-3x_5\},\\ U_2 &= \{(x_1,x_2,x_3,x_4,x_5) : 2x_1-x_2+x_3+4x_4+4x_5 = 0\}.\end{aligned}$$

Is it true that $U_1+U_2=\mathbb{R}^5$?

3. Consider the following space of polynomials:

$$V = \{\mathbf{p}(x)\in\mathbb{R}[x] : \mathbf{p}(1)=0\}.$$

By the Remainder Theorem, $\mathbf{p}(x)$ belongs to V iff it is *divisible* by $x-1$. What is the dimension of $V\cap\mathbb{R}_n[x]$ if $n\geqslant 1$?

4. Consider the following subspaces of $\mathbb{R}^5$:

$$\begin{aligned} V_1 &= \mathscr{L}\{(1,3,-2,2,3),\ (1,4,-3,4,2),\ (2,3,-1,-2,9)\},\\ V_2 &= \mathscr{L}\{(1,3,0,2,1),\ (1,5,-6,6,3),\ (2,5,3,2,1)\}.\end{aligned}$$

Determine the dimensions of V_1, V_2, $V_1\cap V_2$, V_1+V_2.

5. Do the same for

$$\begin{aligned} W_1 &= \{(x_1,x_2,x_3,x_4,x_5) : 2x_1-x_2-x_3 = 0 = x_4-3x_5\},\\ W_2 &= \{(x_1,x_2,x_3,x_4,x_5) : 3x_1-3x_2-x_4 = 0 = 2x_1-x_2-x_3\}.\end{aligned}$$

Referring to the previous exercise, find also the dimensions of $V_1\cap W_1$ and $V_2\cap W_2$.