Notes 15 – Linear Mappings and Matrices

In this lecture, we turn attention to linear mappings that may be neither surjective nor injective. We show that once bases have been chosen, a linear map is completely determined by a matrix. This approach provides the first real justification for the definition of matrix multiplication that we gave in the first lecture.

L15.1 The linear mapping associated to a matrix. First, we point out that any matrix defines a linear mapping.

Lemma. *A matrix $A \in \mathbb{R}^{m,n}$ defines a linear mapping $f\colon \mathbb{R}^n \to \mathbb{R}^m$ by regarding elements of $\mathbb{R}^n$ as column vectors and setting*

$$f(\mathbf{v}) = A\mathbf{v}, \qquad \mathbf{v} \in \mathbb{R}^{n,1}.$$

Proof. The conditions (LM1) and (LM2) are obvious consequences of the rules of matrix algebra. QED

Our preference for column vectors means that an $m \times n$ matrix defines a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, so that m, n are 'swapped over'. Here is an example:

$$A = \begin{pmatrix} 0 & 2 & 4 \\ 3 & 5 & 1 \end{pmatrix} \in \mathbb{R}^{2,3} \quad \text{defines} \quad f\colon \mathbb{R}^3 \to \mathbb{R}^2$$

with

$$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \quad f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

The jth column of A represents the image of the jth element of the canonical basis of $\mathbb{R}^{n,1}$.

This example shows that at times we can use A and f interchangeably. But there is a subtle difference: in applying f, we are allowed to represent elements of $\mathbb{R}^n$ and $\mathbb{R}^m$ by row vectors. Thus it is also legitimate to write

$$f(1,0,0) = (0,3), \quad f(0,1,0) = (2,5), \quad f(0,0,1) = (4,1),$$

and more memorably,

$$f(x_1, x_2, x_3) = (2x_2 + 4x_3,\ 3x_1 + 5x_2 + x_3). \tag{1}$$

The last equation shows us how to pass from the *rows* of A to the definition of f. It turns out that any linear mapping $f\colon \mathbb{R}^n \to \mathbb{R}^m$ has a form analogous to (1), from which we can construct the rows of an associated matrix.

L15.2 The matrix associated to a linear mapping. Let V, W be vector spaces.

Lemma. *Let V be a vector space with basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. A linear mapping $f\colon V \to W$ is completely determined by vectors $f(\mathbf{v}_1), \dots, f(\mathbf{v}_n)$, which (in order to define f) can be assigned arbitrarily.*

Proof. Suppose that $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ is a basis of V. Then any element $\mathbf{v}$ of V can be written in a *unique* way as $a_1\mathbf{v}_1+\cdots+a_n\mathbf{v}_n$, and

$$f(\mathbf{v}) = f(a_1\mathbf{v}_1+\cdots+a_n\mathbf{v}_n) = a_1 f(\mathbf{v}_1)+\cdots+a_n f(\mathbf{v}_n). \tag{2}$$

Thus, $f(\mathbf{v})$ is determined by the n images $f(\mathbf{v}_i)$ of the basis elements. Moreover, *any* choice of n such vectors allows us to define a linear mapping f by means of (2). QED

We have seen that a matrix determines a linear mapping between vector spaces, namely from $\mathbb{R}^n$ to $\mathbb{R}^m$. The Lemma allows us to go backwards and associate a matrix to *any* linear mapping from V to W, once we have chosen bases

$$\{\mathbf{v}_1,\cdots,\mathbf{v}_n\}, \qquad \{\mathbf{w}_1,\cdots,\mathbf{w}_m\} \tag{3}$$

of V and W (we are assuming that $\dim V = n$ and $\dim W = m$).

Definition. *Let $g\colon V\to W$ be a linear mapping. The* matrix *of g with respect to the bases (3) is the matrix $B\in\mathbb{R}^{m,n}$ (also written M_g or $M(g)$ if the choice (3) is understood) whose jth column gives the coefficients of $g(\mathbf{v}_j)$ in terms of $\mathbf{w}_1,\ldots,\mathbf{w}_m$.*

If $B=(b_{ij})$, then we are asserting that

$$g(\mathbf{v}_j) = b_{1j}\mathbf{w}_1+\cdots+b_{mj}\mathbf{w}_m = \sum_{i=1}^{m} b_{ij}\mathbf{w}_i, \qquad \text{for each } j=1,\ldots,n. \tag{4}$$

By the previous lemma, g is completely determined by B. 𝔚𝔞𝔯𝔫𝔦𝔫𝔤: In this definition, we are really thinking of the bases (3) as *ordered* sets.

To recover our previous correspondence, we need to take $V=\mathbb{R}^n$ and $W=\mathbb{R}^m$ with their *canonical bases.* For let $A\in\mathbb{R}^{m,n}$. If $f\colon\mathbb{R}^n\to\mathbb{R}^m$ is the linear mapping defined by setting $f(\mathbf{v})=A\mathbf{v}$ and we choose (3) to be the canonical bases, then $B=M_f=A$.

Example. We are perfectly at liberty to apply the Definition to the same vector space with different bases. Let $V=W=\mathbb{R}_2[x]$. Choose the basis $\{1,\ x+1,\ (x+1)^2\}$ for V and the basis $\{1,x,x^2\}$ for W. Let D be the linear mapping defined by differentiation: $D\mathbf{p}=\mathbf{p}'$. Then the matrix of D with respect to the chosen bases is

$$A = \begin{pmatrix} 0 & 1 & 2\\ 0 & 0 & 2\\ 0 & 0 & 0\end{pmatrix};$$

the null column tells us that $D(1)=0$ and the null row tells us that $\operatorname{Im} D$ consists of polynomials of degree no greater than 1.

L15.3 Compositions and products. With the link between linear mappings and matrices now established, we shall see that composition of matrices corresponds to the product of matrices. Suppose that $B\in\mathbb{R}^{m,n}$, $A\in\mathbb{R}^{n,p}$, and consider the associated linear mappings

$$\mathbb{R}^{m,1} \overset{g}{\longleftarrow} \mathbb{R}^{n,1} \overset{f}{\longleftarrow} \mathbb{R}^{p,1}$$

defined by $f(\mathbf{u})=A\mathbf{u}$ and $g(\mathbf{v})=B\mathbf{v}$. (It is easier to understand what follows by writing the mappings from right to left.) The composition $g\circ f$ is obviously

$$BA\mathbf{u} \leftarrow A\mathbf{u} \leftarrow \mathbf{u},$$

and is therefore associated to the matrix BA.

More generally, given vector spaces U, V, W and linear mappings

$$W \stackrel{g}{\longleftarrow} V \stackrel{f}{\longleftarrow} U, \tag{5}$$

choose bases for each of U, V, W, and let M_f, M_g be the associated matrices (the same basis of V being used for both matrices). Then we state without proof the

Proposition. *Let $h = g \circ f$ be the composition (5). Then $M_h = M_g M_f$.*

This result is especially useful in the case of a single vector space V of dimension n, and a linear mapping $f\colon V \to V$. Such a linear mapping (between the same vector space) is called a *linear transformation* or *endomorphism.* In these circumstances, we can fix the same basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V, and consider compositions of f with itself:

> **Example.** Define $f : \mathbb{R}^3 \to \mathbb{R}^3$ by $f(\mathbf{e}_1) = \mathbf{e}_2$, $f(\mathbf{e}_2) = \mathbf{e}_3$ and $f(\mathbf{e}_3) = \mathbf{e}_1$. Check that the matrix $A = M_f$ (taken with respect to the canonical basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$) satisfies $A^3 = I_3$.

L15.4 Nullity and rank. Important examples of subspaces are provided by the

Lemma. *Let $g\colon V \to W$ be a linear mapping. Then*
(i) $g^{-1}(\mathbf{0})$ is a subspace of V,
(ii) $\operatorname{Im} g$ is a subspace of W.

Proof. We shall only prove (ii). If $\mathbf{w}_1, \mathbf{w}_2 \in \operatorname{Im} g$ then we may write $\mathbf{w}_1 = g(\mathbf{v}_1)$ and $\mathbf{w}_2 = g(\mathbf{v}_2)$ for some $\mathbf{v}_1, \mathbf{v}_2 \in V$. If $a \in F$ then

$$a\mathbf{w}_1 + \mathbf{w}_2 = ag(\mathbf{v}_1) + g(\mathbf{v}_2) = g(a\mathbf{v}_1 + \mathbf{v}_2),$$

and so $a\mathbf{w}_1 + \mathbf{w}_2 \in \operatorname{Im} g$. Part (i) is similar. QED

> **Example.** In the case of a linear mapping $f\colon \mathbb{R}^n \to \mathbb{R}^m$ defined by $f(\mathbf{v}) = A\mathbf{v}$ with $A \in \mathbb{R}^{m,n}$,
>
> $$f^{-1}(\mathbf{0}) = \{\mathbf{v} \in \mathbb{R}^{n,1} : A\mathbf{v} = \mathbf{0}\}$$
>
> is the solution space of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. We already know that this is a subspace and we labelled it $\operatorname{Ker} A$. On the other hand, the image of f is generated by the vectors $f(\mathbf{e}_i)$ that are the columns of A:
>
> $$\operatorname{Im} f = \mathscr{L}\{f(\mathbf{e}_1), \ldots, f(\mathbf{e}_n)\} = \mathscr{L}\{\mathbf{c}_1 \ldots, \mathbf{c}_n\} = \operatorname{Col} A.$$
>
> It follows that the dimension of $\operatorname{Im} f$ equals the rank, $r(A)$, the common dimension of $\operatorname{Row} A$ and $\operatorname{Col} A$.

In view of this key example, we state the

Definition. *(i) The* kernel *of an arbitrary linear mapping $g\colon V \to W$ is the subspace $g^{-1}(\mathbf{0})$, more usually written* $\operatorname{Ker} g$ *or* $\ker g$. *Its dimension is called the* nullity *of g.*
(ii) The dimension of $\operatorname{Im} g$ *is called the* rank *of g.*

Rank-Nullity Theorem. *Given an arbitrary linear mapping $g\colon V \to W$,*

$$\dim V = \dim(\operatorname{Ker} g) + \dim(\operatorname{Im} g).$$

This important result can also be expressed in the form

$$\operatorname{nullity}(g) + \operatorname{rank}(g) = n,$$

n being the dimension of the space of 'departure'.

Proof. By choosing bases for V and W, we may effectively replace g by a linear mapping $f:\mathbb{R}^n \to \mathbb{R}^m$. But in this case, the previous example shows that $\mathrm{Ker}\, f = \mathrm{Ker}\, A$ and $\mathrm{Im}\, f = \mathrm{Col}\, A$. We know that $\dim(\mathrm{Col}\, A) = \dim(\mathrm{Row}\, A) = r(A)$, and (essentialy by (RC2)),

$$\dim(\mathrm{Ker}\, A) = n - r(A).$$

The result follows. QED

The following result is for class discussion:

Corollary. *Given a linear mapping $g:V \to W$ with $\dim V = n$ and $\dim W = m$,*

$$\begin{array}{lcl} g \text{ is injective} & \Leftrightarrow & \mathrm{rank}(g) = n \\ g \text{ is surjective} & \Leftrightarrow & \mathrm{rank}(g) = m \\ g \text{ is bijective} & \Leftrightarrow & \mathrm{rank}(g) = m = n. \end{array}$$

L15.5 Further exercises.

1. Let $f:\mathbb{R}^4 \to \mathbb{R}^3$ be the linear mapping with

$$f(x_1,x_2,x_3,x_4) = (x_1,\ x_1{+}x_2,\ x_1{+}x_2{+}x_3).$$

Write down that associated matrix A, and find $\mathbf{v} \in \mathbb{R}^4$ such that $f(\mathbf{v}) = (0,1,1)$.

2. Let $f:\mathbb{R}^3 \to \mathbb{R}^2$ and $g:\mathbb{R}^2 \to \mathbb{R}^3$ be the two mappings:

$$f(x,y,z) = (x+y,\ x+y-z), \qquad g(s,t) = (3s-t,\ 3t-s,\ s).$$

(i) Complete: $g(f(x,y,z)) = (2x{+}2y{+}z,\quad ?\quad)$
(ii) Find the matrices $M_f, M_g, M_{g\circ f}$ with respect to the canonical bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, and check that $M_{g\circ f} = M_g M_f$.

3. Let $V = W = \mathbb{R}_2[x]$, and let D be the linear mapping given by $D(\mathbf{p}(x)) = \mathbf{p}'(x)$. Find the matrix M_D with respect to the bases: $\{1, x, x^2\}$ for V and $\{1,\ x{+}1,\ (x{+}1)^2\}$ for W.

4. Let $f:\mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$\mathrm{Ker}\, f = \mathscr{L}\{(1,0,0),(0,1,0)\} \quad \text{and} \quad \mathrm{Im}\, f \subseteq \mathscr{L}\{(0,1,0),(0,0,1)\}.$$

Find all possible matrices M_f associated to f with respect to the canonical basis of $\mathbb{R}^3$.

5. Find bases for the kernel and the image of each of the following linear mappings:

$$f:\mathbb{R}^3 \to \mathbb{R}^{2,2}, \quad f(x,y,z) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix},$$

$$g:\mathbb{R}^{2,2} \to \mathbb{R}^4, \quad g\begin{pmatrix} x & y \\ z & t \end{pmatrix} = (x-2y,\ x-2z,\ y+t,\ x+2t).$$

6. Let $f:\mathbb{R}^{3,1} \to \mathbb{R}^{3,1}$ be the linear transformation defined by $A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 3 \\ 3 & 5 & 1 \end{pmatrix}$.
(i) Find a vector $\mathbf{v}_1$ such that $\mathrm{Ker}\, f = \mathscr{L}\{\mathbf{v}_1\}$.
(ii) *Choose* $\mathbf{v}_2, \mathbf{v}_3$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbb{R}^{3,1}$.
(iii) Check that $\{f(\mathbf{v}_2), f(\mathbf{v}_3)\}$ is a basis of the subspace $\mathrm{Im}\, f$ (it always will be!).